The Schützenberger product for Syntactic Spaces

Mai Gehrke, Daniela Petrişan and Luca Reggio

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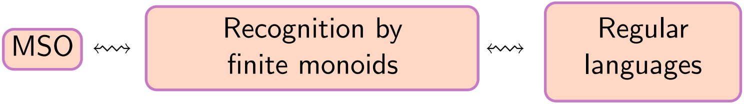
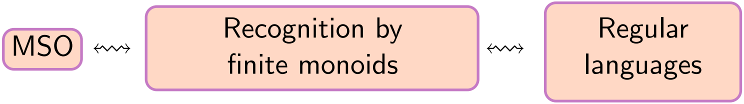
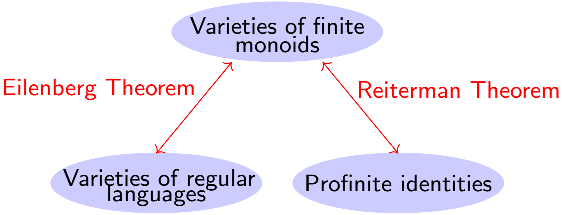


E.g. profinite equations for the regular fragment of $AC^0 = FO(\mathcal{N})$
(Barrington, Compton, Straubing and Thérien 1990)





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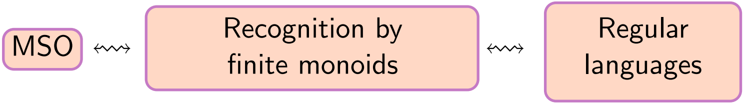

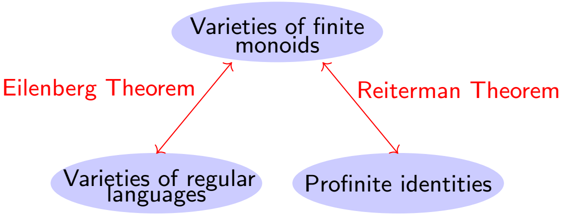

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- $\widehat{A^*}$ is the dual Stone space of $Reg(A^*)$ (Birkhoff 1937, Almeida 1994, Pippenger 1997, ...);
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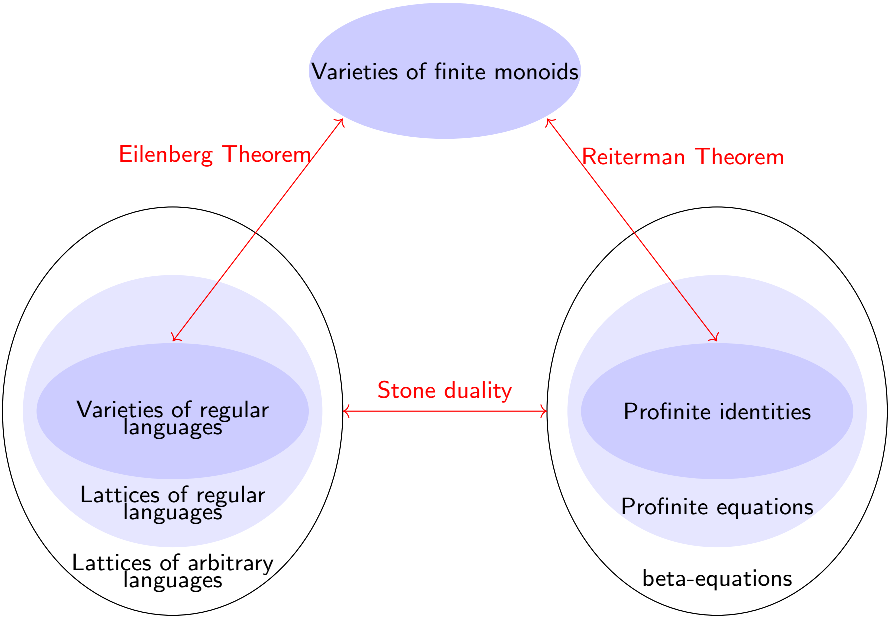
Varieties of finite monoids
Eilenberg Theorem
Reiterman Theorem
Varieties of regular languages
Lattices of regular languages
Lattices of arbitrary languages
Stone duality
Profinite identities
Profinite equations
beta-equations

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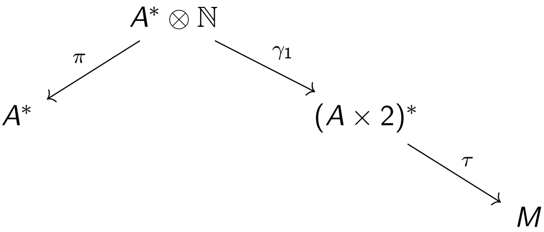

where

$$A^* \otimes \mathbb{N} := \{(w, i) \in A^* \times \mathbb{N} \mid i < |w|\}$$

is the set of **words with a marked spot**.

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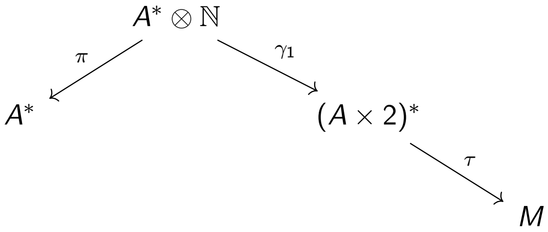

$$L_{\exists x.\Phi(x)} = \pi[\gamma_1^{-1}(L_{\Phi(x)})]$$

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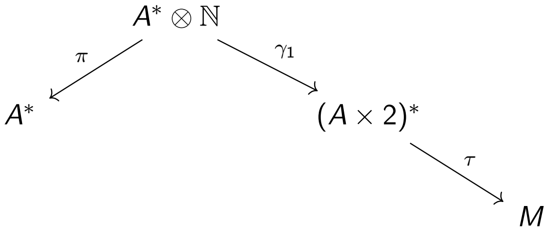

$$A^* \longrightarrow \mathcal{P}_{fin}(M)$$

Definition

If M is a monoid, then $\Diamond M$ is a bilateral semidirect product: it has $\mathcal{P}_{fin}(M) \times M$ as underlying set and the multiplication is given by

$$(S, m) * (T, n) := (S \cdot n \cup m \cdot T, m \cdot n).$$

Lemma

If M recognises $L_{\Phi(x)}$, then $\Diamond M$ recognises $L_{\exists x.\Phi(x)}$.

$$L_{\Phi(x)} \rightsquigarrow L_{\exists x.\Phi(x)}$$
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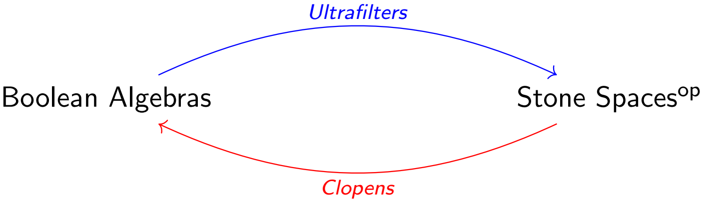
Consider a finite monoid M and a morphism $\mu\colon A^* \twoheadrightarrow M$. Then the Boolean algebra of A^*-languages recognised by μ is isomorphic to the power-set algebra $\mathcal{P}(M)$.

$$\frac{\mu\colon A^* \twoheadrightarrow M}{\mathcal{P}(A^*) \hookleftarrow \mathcal{P}(M)\colon \mu^{-1}}$$

There is a correspondence between monoid quotients and embeddings as power-set subalgebras closed under quotients. This is a special case of a more general phenomenon, namely **Stone duality**.

Stone duality for Boolean algebras



Stone spaces = Zero-dimensional compact Hausdorff spaces

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Stone duality for Boolean algebras

$$\text{Boolean Algebras} \underset{\textit{Clopens}}{\overset{\textit{Ultrafilters}}{\rightleftarrows}} \text{Stone Spaces}^{\text{op}}$$

Stone spaces = Zero-dimensional compact Hausdorff spaces

An example is

$$\mathcal{P}(A^*) \underset{\textit{Clopens}}{\overset{\textit{Ultrafilters}}{\rightleftarrows}} \beta(A^*) = \text{Stone-Čech compactification of } A^*$$

The recognising objects

Let $\mathcal{B} \hookrightarrow \mathcal{P}(A^*)$ be any Boolean algebra of languages closed under quotients. Dually, we have a continuous surjection $\tau\colon \beta(A^*) \twoheadrightarrow X_{\mathcal{B}}$.

$$\begin{array}{ccc} \beta(A^*) & \xrightarrow{\quad \tau \quad} & X_{\mathcal{B}} \\ \uparrow & & \\ A^* & & \end{array}$$

NOTE: A^* is a monoid which acts continuously (on the left and on the right) on $\beta(A^*)$. Further, A^* is dense in $\beta(A^*)$.

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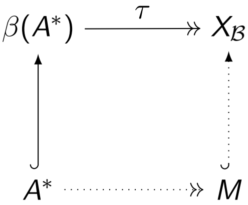

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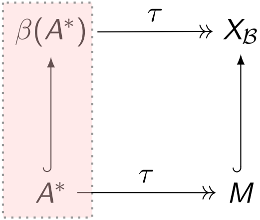

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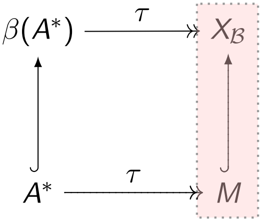

NOTE: M is a monoid which acts continuously (on the left and on the right) on X. Further, M is dense in X.

The recognising objects

Definition

A **Stone space with an internal monoid** is a pair (X, M) where

- X is a Stone space
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- the biaction of M on itself extends to a biaction of M on X with continuous components

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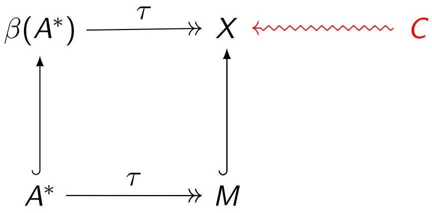

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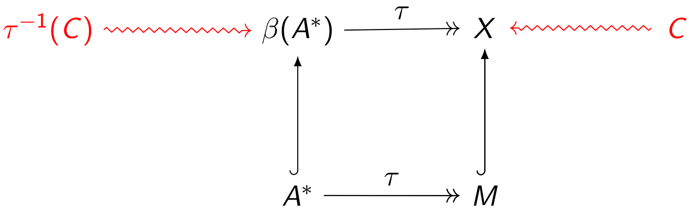

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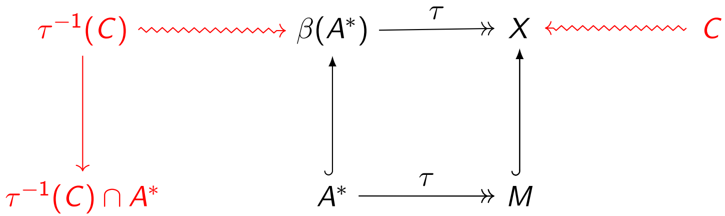

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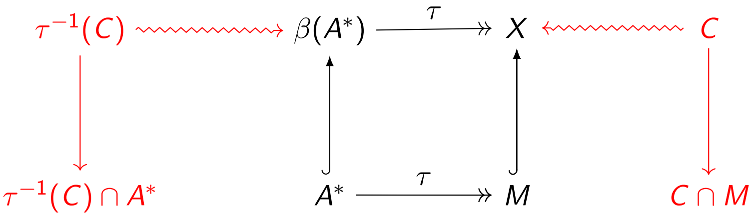

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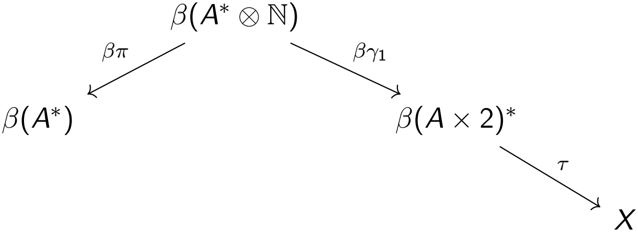

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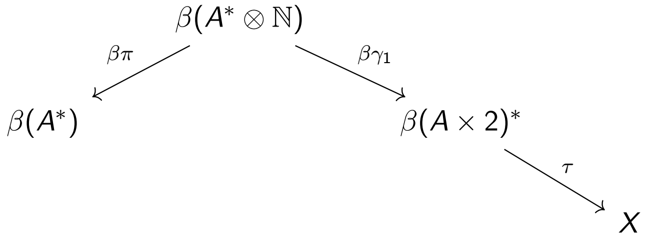

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Let X be any topological space. Then its **Vietoris hyperspace** has $\mathcal{V}(X) = \{K \subseteq X \mid K \text{ is closed}\}$ as underlying set, and its topology is the generated by the sets of the form

$$\Box U = \{K \in \mathcal{V}(X) \mid K \subseteq U\}, \ \Diamond U := \{K \in \mathcal{V}(X) \mid K \cap U \neq \emptyset\}$$

for U an open subset of X.

Theorem

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- Notion of recognition in the topological setting
- $\lozenge$ as dual construction to $\exists$

What I have not presented (look up in the paper!):

- The binary product
- Ultrafilter equations

Future directions:

- "Good" basis of equations
- Connections with generalization of block product (Krebs et al.)

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