Approximate Span Programs

New Techniques for Designing Quantum Algorithms

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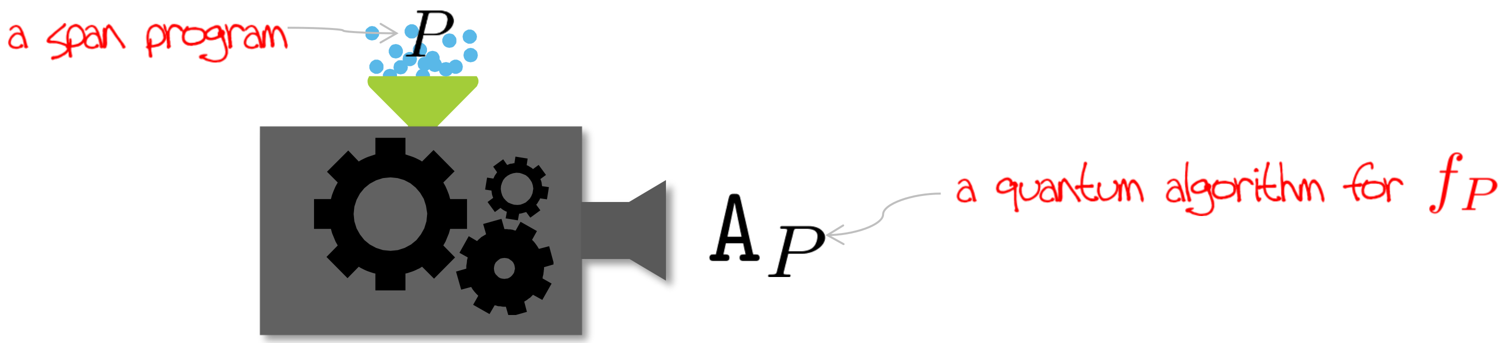
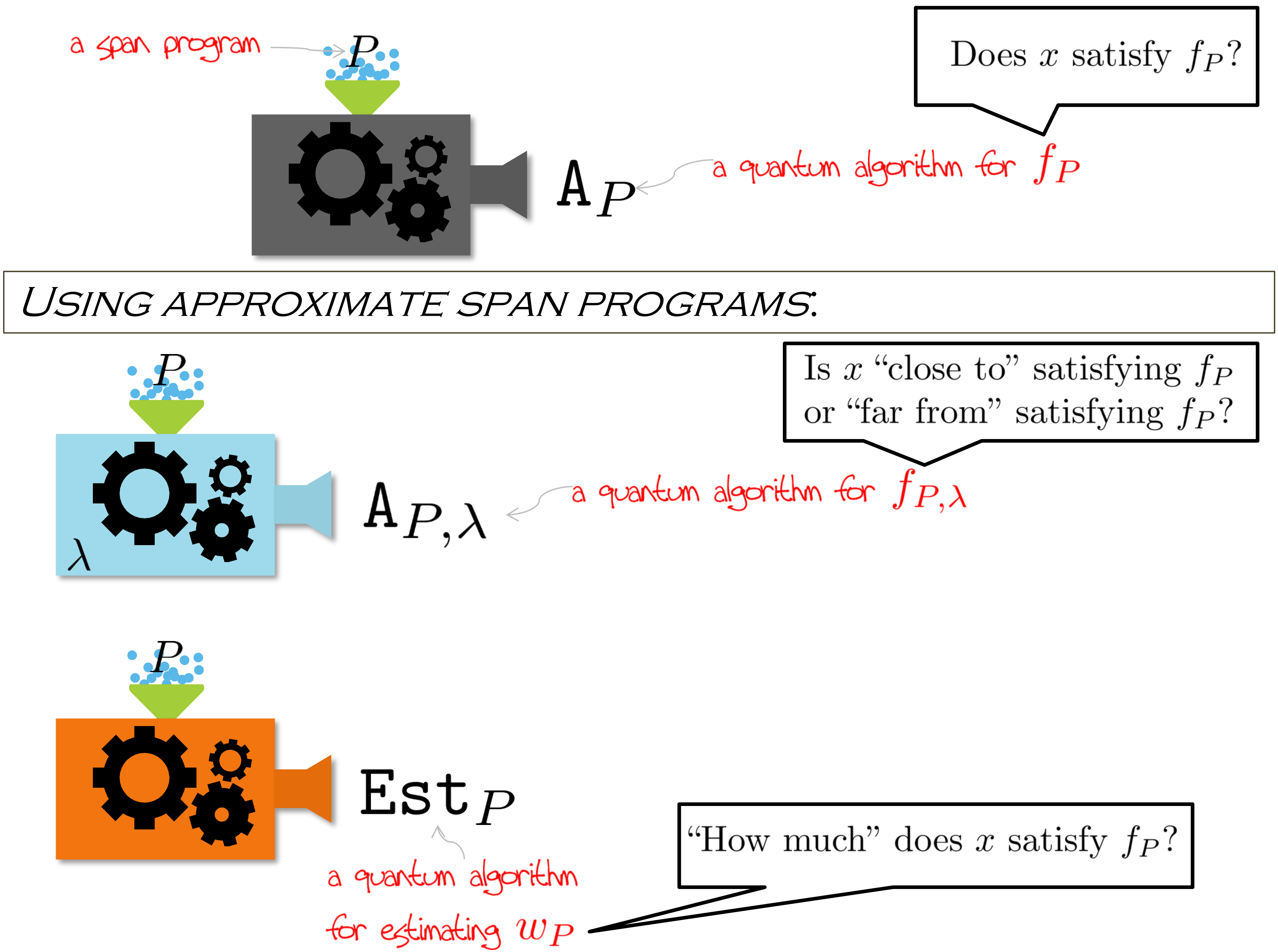
Stacey Jeffery

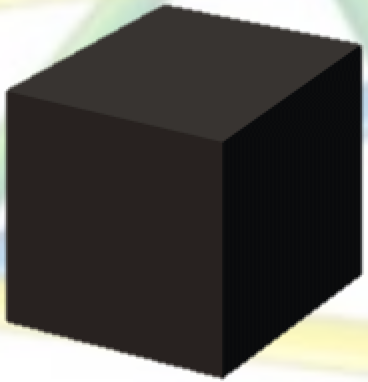


a span program
P
$\mathtt{A}_P$
a quantum algorithm for f_P



a span program
P
Does x satisfy f_P?
$\mathtt{A}_P$
a quantum algorithm for f_P
USING APPROXIMATE SPAN PROGRAMS:
P
Is x "close to" satisfying f_P or "far from" satisfying f_P?
λ
$\mathtt{A}_{P,\lambda}$
a quantum algorithm for $f_{P,\lambda}$
P
$\mathtt{Est}_P$
"How much" does x satisfy f_P?
a quantum algorithm for estimating w_P

Quantum Query Complexity



Quantum Query Complexity

$x \in \{0,1\}^n$

quantum query complexity of an algorithm:

$Q(\mathcal{A}) =$ number of oracle calls made by $\mathcal{A}$

(bounded error) quantum query complexity of a function:

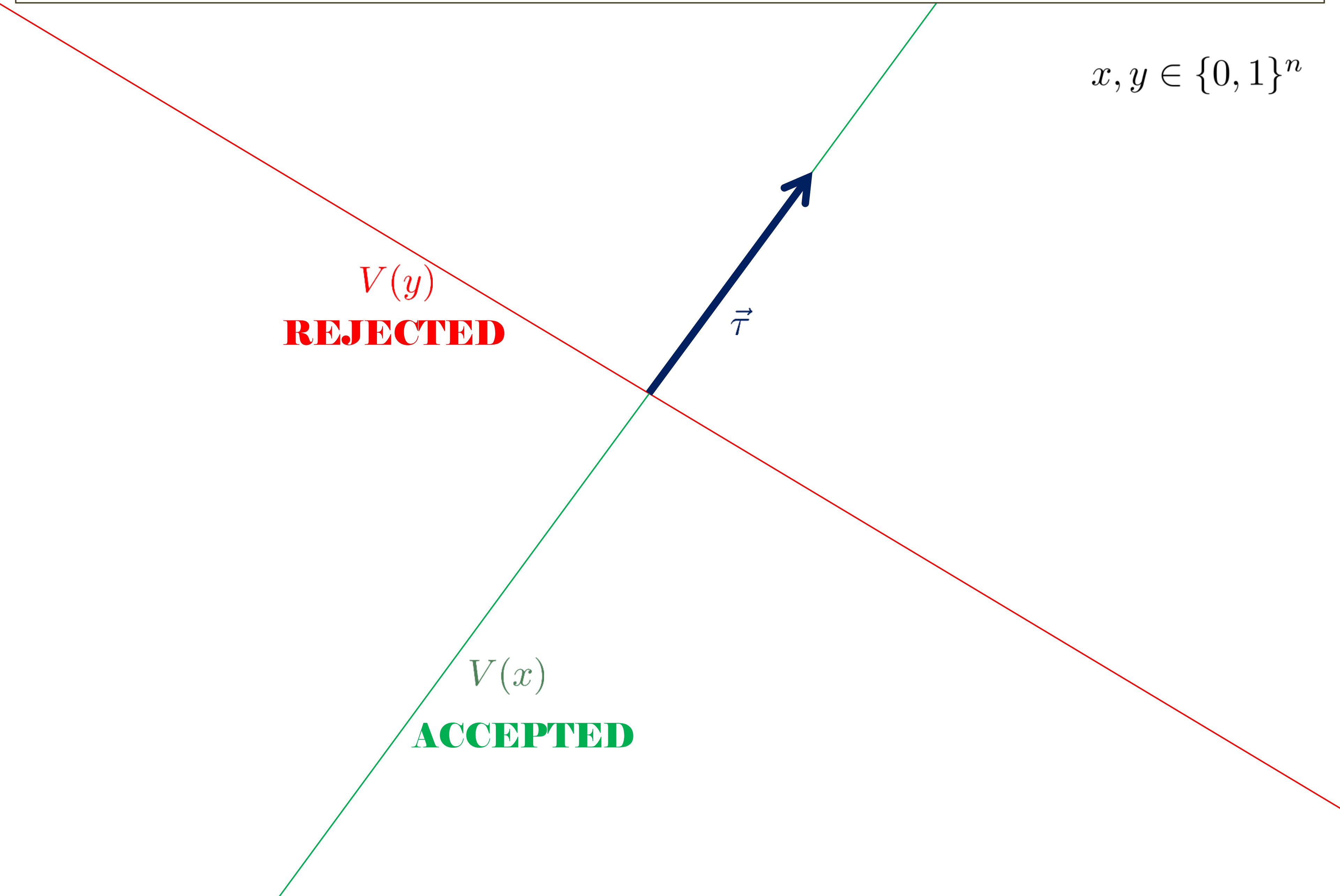
$Q(f) =$ minimum $Q(\mathcal{A})$ over all bounded error algorithms for f, $\mathcal{A}$

bounded error: for any input, algorithm must be correct with probability $\geq \frac{2}{3}$

Span Programs

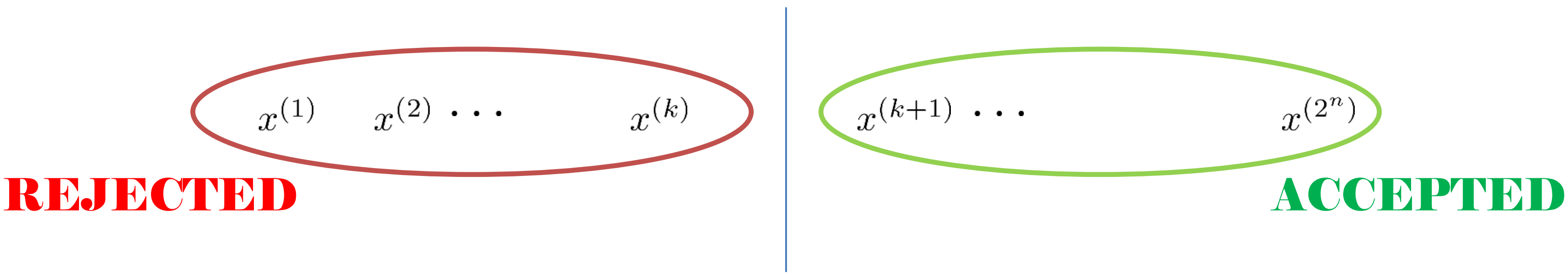
Span Programs

$x, y \in \{0,1\}^n$



Span Programs

$$\{0,1\}^n = \{x^{(1)}, \ldots, x^{(2^n)}\}$$



$$f_P(x) = \begin{cases} 1 & \text{if } x \text{ is accepted by the span program} \\ 0 & \text{if } x \text{ is rejected by the span program} \end{cases}$$

The span program decides f_P.

Theorem: For any span program P, there is a quantum algorithm that decides f_P in query complexity equal to the *span program complexity*. [Reichardt '09]

Span Programs

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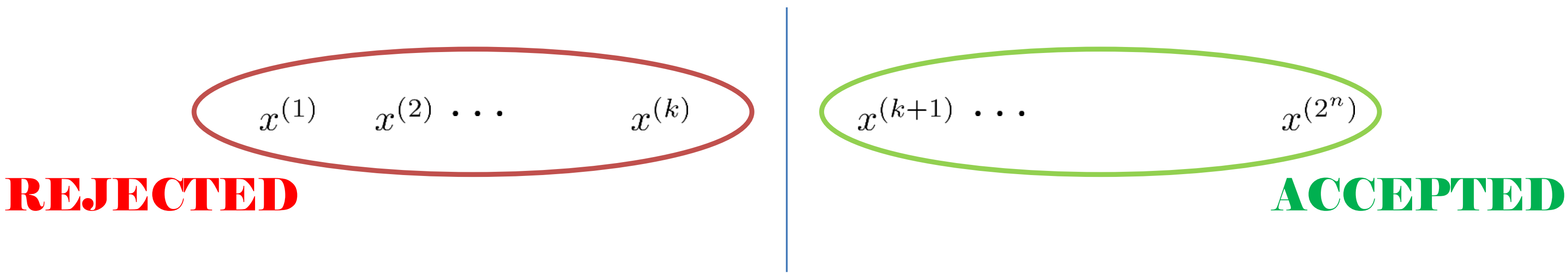

x is accepted $\Leftrightarrow$ x has a positive witness $\vec{w}$

positive witness size of x: $w_+(x) = \min_{\vec{w}} \|\vec{w}\|^2$

x is rejected $\Leftrightarrow$ x has a negative witness $\vec{v}$

negative witness size of x: $w_-(x) = \min_{\vec{v}} \|\vec{v}\|^2$

$W_+ = \max_x w_+(x)$ $\qquad$ $W_- = \max_x w_-(x)$

Theorem: For any span program P, there is a quantum algorithm that decides f_P in query complexity equal to the *span program complexity*. [Reichardt '09]

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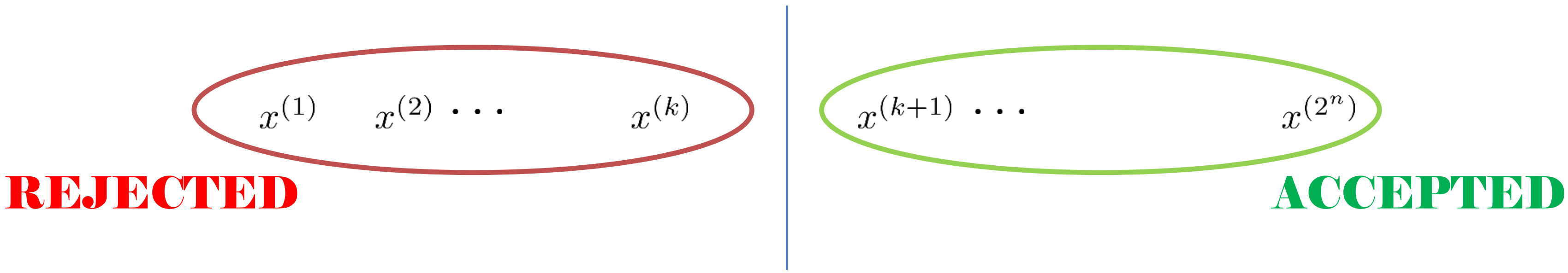

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Theorem: For any span program P, there is a quantum algorithm that decides f_P in query complexity equal to $O(\sqrt{W_+W_-})$.

[Reichardt '09]

Span Programs

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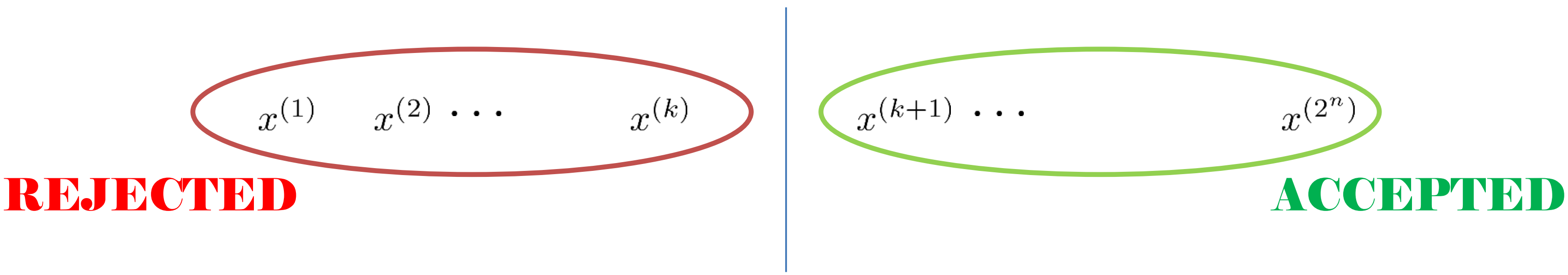

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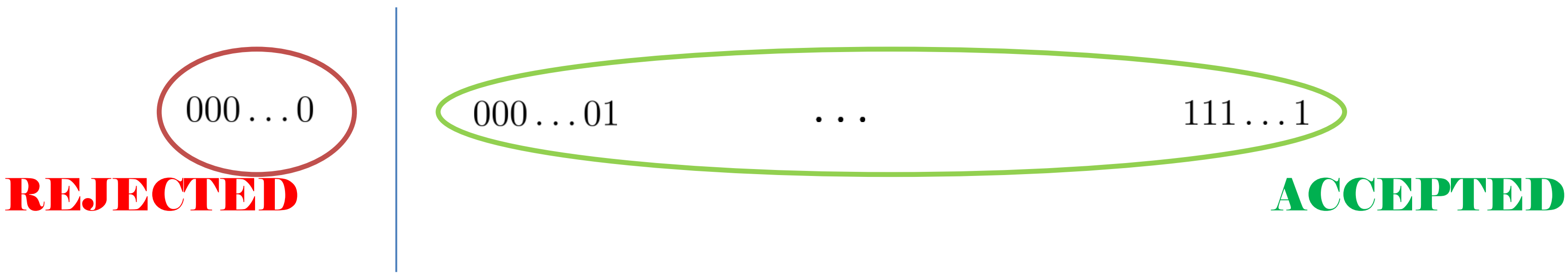
$D \subseteq \{0,1\}^n$ $\quad W_+ = \max_{x\in D} w_+(x)$ $\quad W_- = \max_{x\in D} w_-(x)$

Theorem: For any span program P, there is a quantum algorithm that decides $f_P|_D$ in query complexity equal to $O(\sqrt{W_+W_-})$.

[Reichardt '09]

Example: OR

$$\{0,1\}^n = \{x^{(1)}, \ldots, x^{(2^n)}\}$$



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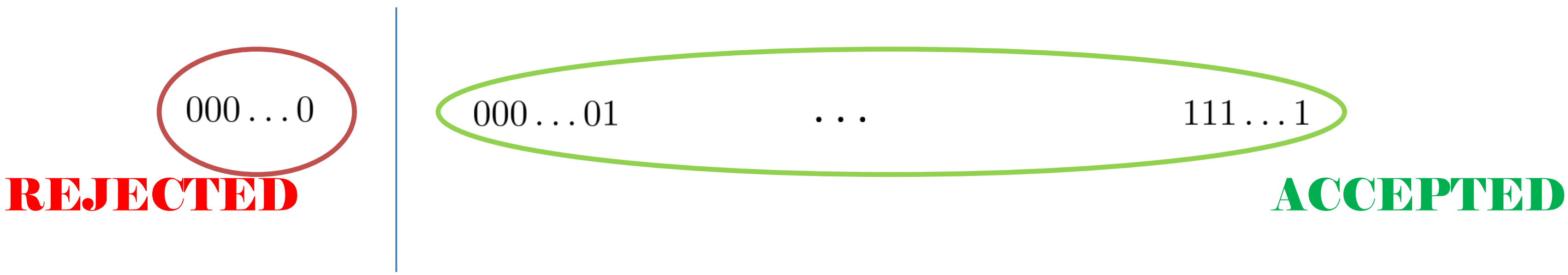

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Example: OR

$$\{0,1\}^n = \{x^{(1)}, \ldots, x^{(2^n)}\}$$

$000\ldots0$

REJECTED

$000\ldots01$ $\quad\ldots\quad$ $111\ldots1$

ACCEPTED

x is accepted $\Leftrightarrow$ x has a positive witness $\vec{w}$

positive witness size of x: $w_+(x) = \frac{1}{|x|}$

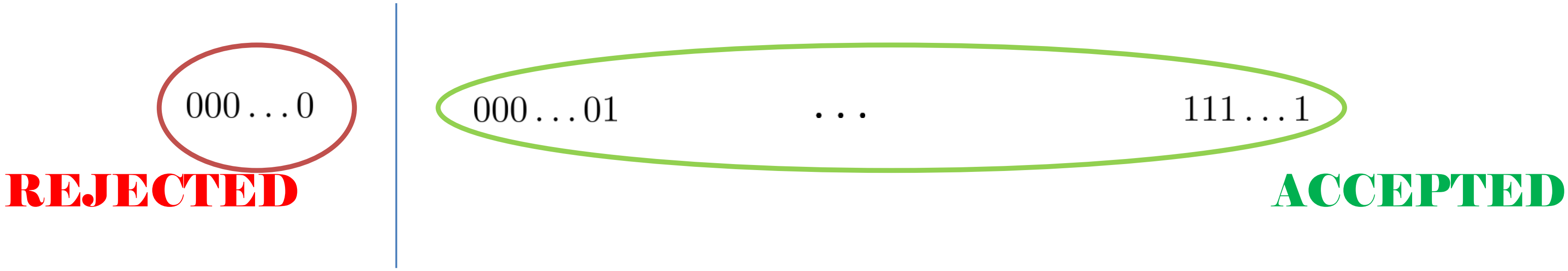
x is rejected $\Leftrightarrow$ x has a negative witness $\vec{v}$

negative witness size of x: $w_-(x) = n$

$W_+ = \max_x w_+(x)$ $\qquad$ $W_- = \max_x w_-(x)$

Example: OR

$$\{0,1\}^n = \{x^{(1)}, \ldots, x^{(2^n)}\}$$



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$$W_+ = \max_x w_+(x) = 1 \qquad W_- = \max_x w_-(x)$$

Example: OR

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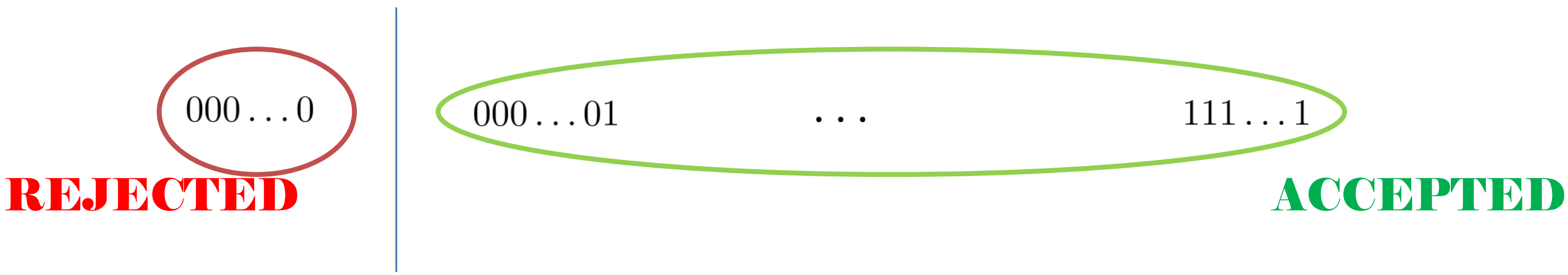

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$$W_+ = \max_x w_+(x) = 1 \qquad W_- = \max_x w_-(x) = n$$

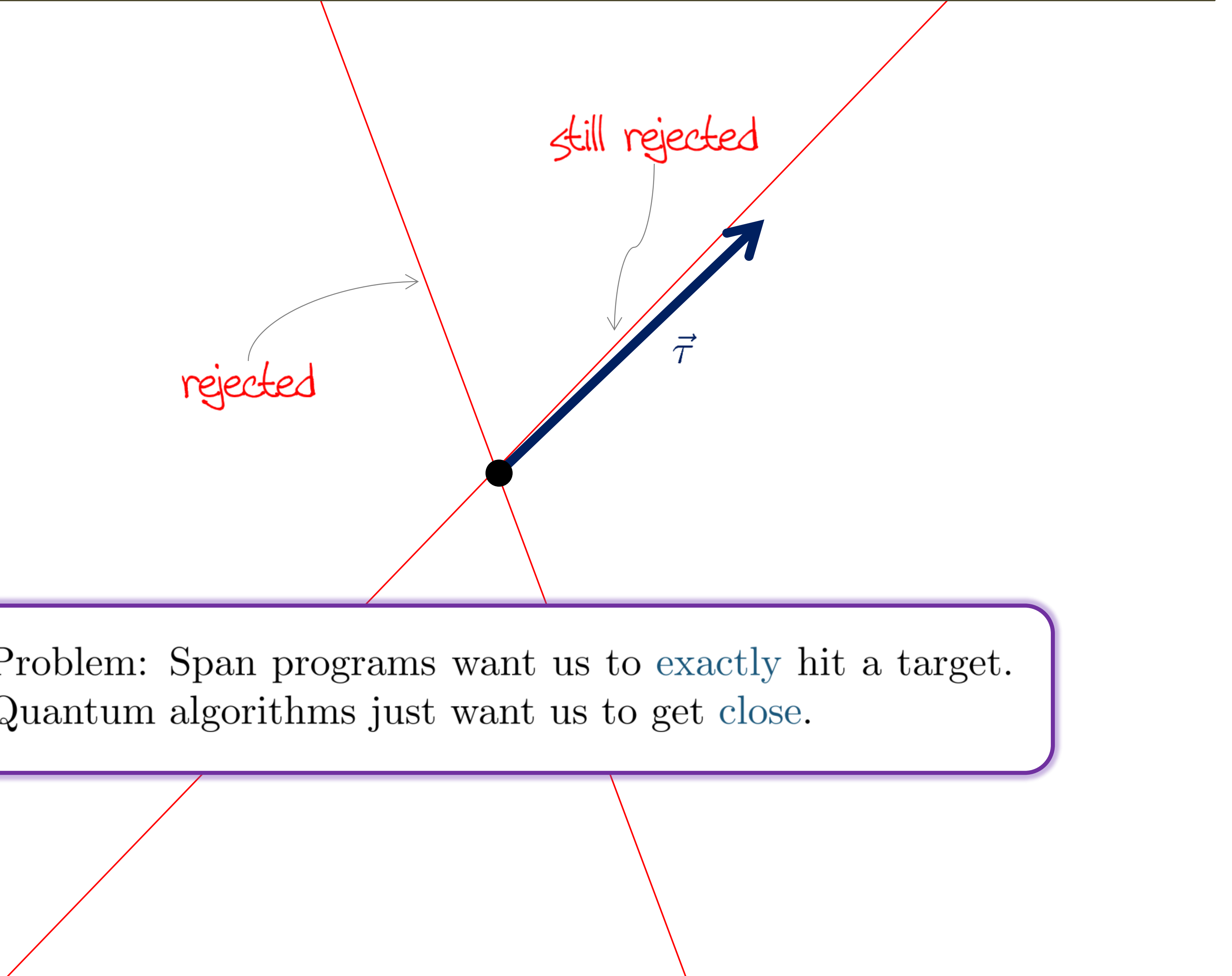
Span program complexity: $\sqrt{W_+ W_-} = \sqrt{n}$

Span Programs = Bounded Error Quantum Query Complexity

For any decision problem f:

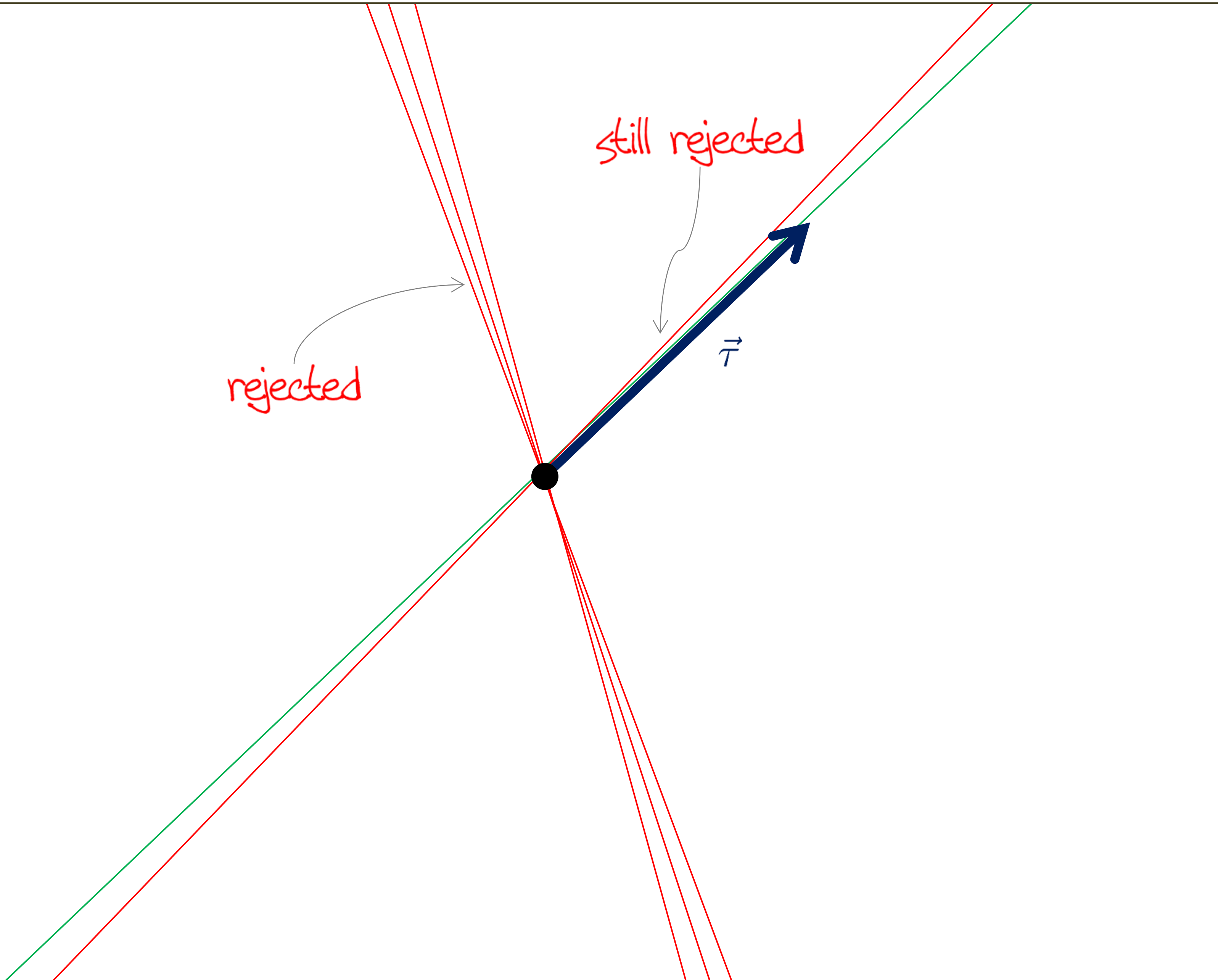
minimumum complexity of a span program that decides f	$=$	bounded error quantum query complexity of f

Exactness of Span Programs



Problem: Span programs want us to exactly hit a target.
Quantum algorithms just want us to get close.

Exactness of Span Programs



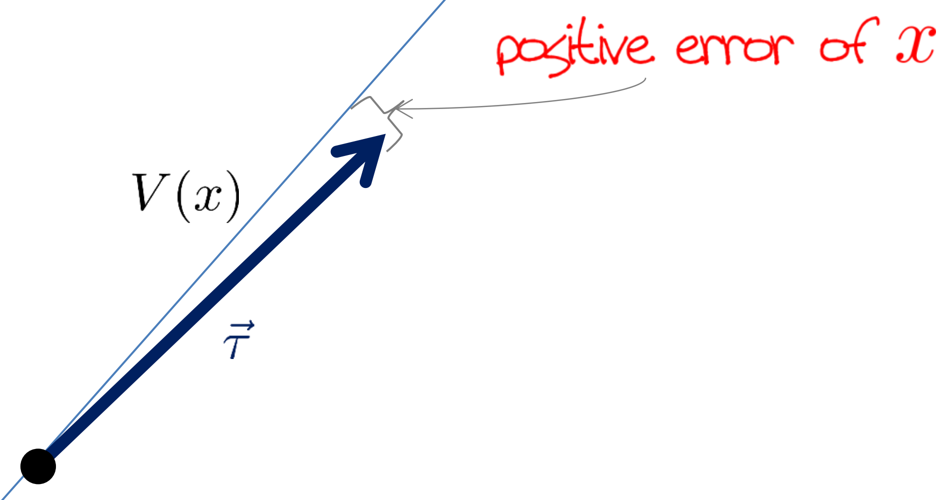
Approximate Span Programs

Approximate Span Programs

Regime 1: span program accepts x if $V(x)$ is close to the target

i.e., accept if x has small positive error

Regime 2: span program rejects x if x has small negative error



When should we say that a span program accepts an input?

Approximate Span Programs

Regime 1: span program accepts x if $V(x)$ is close to the target

i.e., accept if x has small positive error

$\Leftrightarrow$ Only reject if x has a small negative witness

Regime 2: span program rejects x if x has small negative error

$\Leftrightarrow$ Only accept if x has a small positive witness

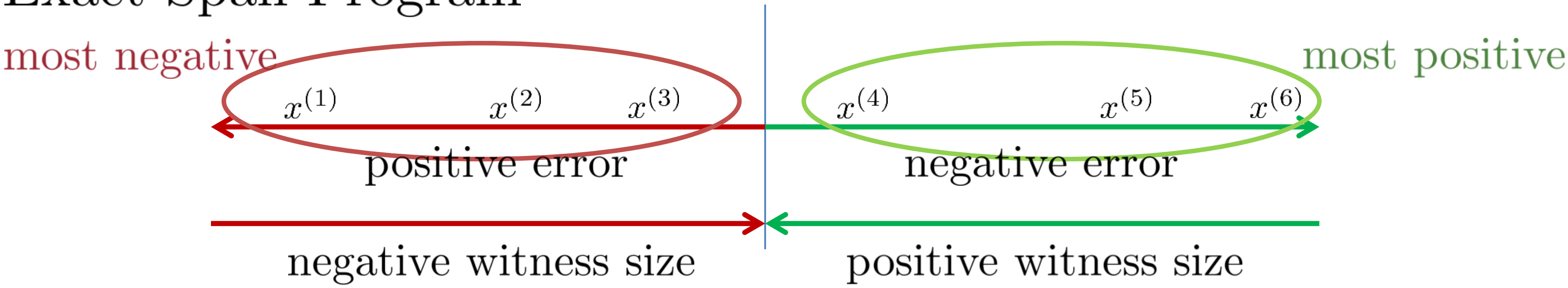
Theorem:

$$\text{positive error of } x = \frac{1}{w_-(x) = \text{negative witness size of } x}$$

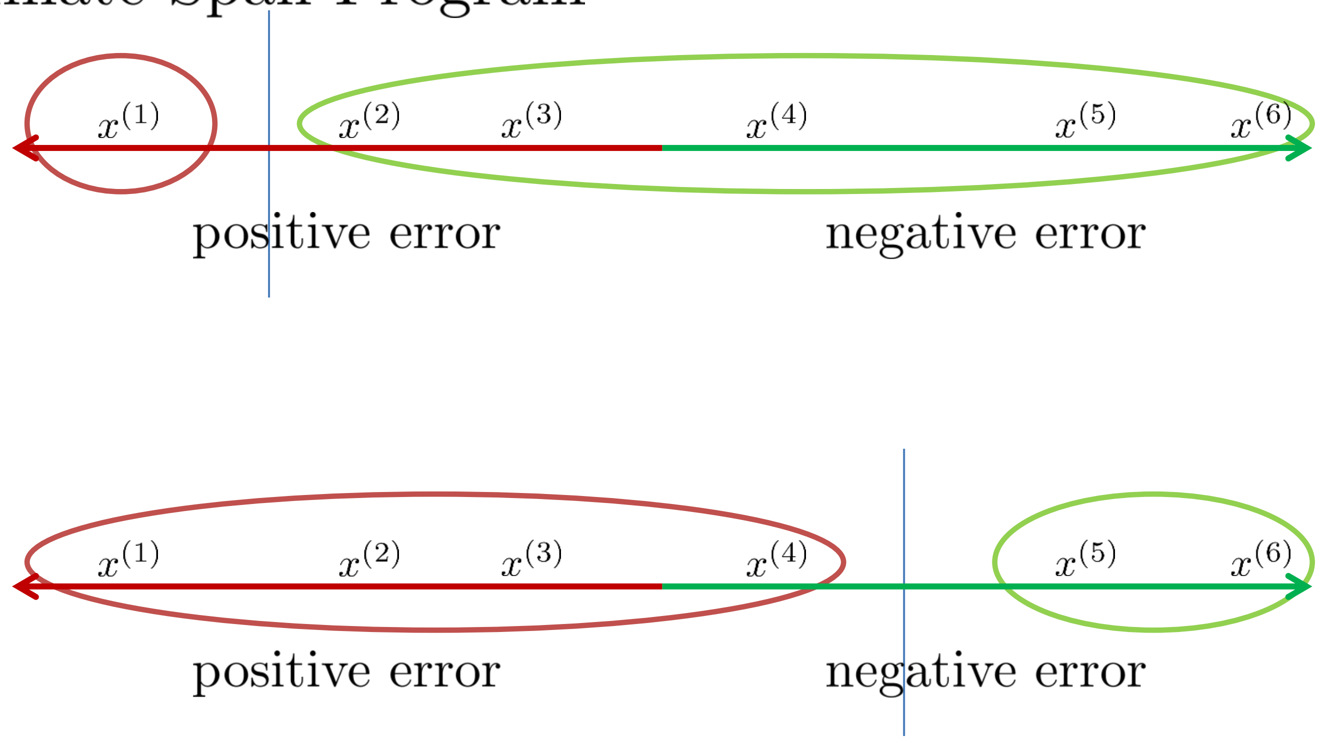
$$\text{negative error of } x = \frac{1}{w_+(x) = \text{positive witness size of } x}$$

Approximate Span Programs

Exact Span Program



Approximate Span Program



Approximate Span Program Algorithm

A span program approximates f if $\frac{\max_{x:f(x)=1} w_+(x)}{\min_{x:f(x)=0} w_+(x)} < \lambda$ for constant $\lambda < 1$.

Span program that *approximates f* with complexity C_f

→

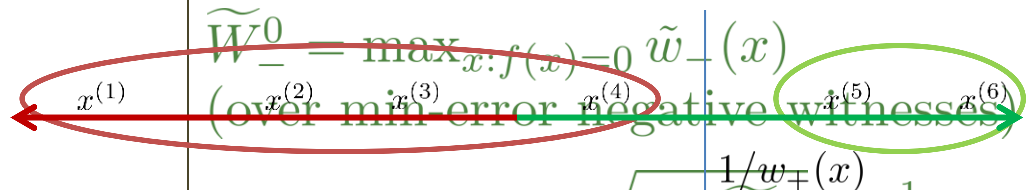
Quantum algorithm for f with query complexity C_f

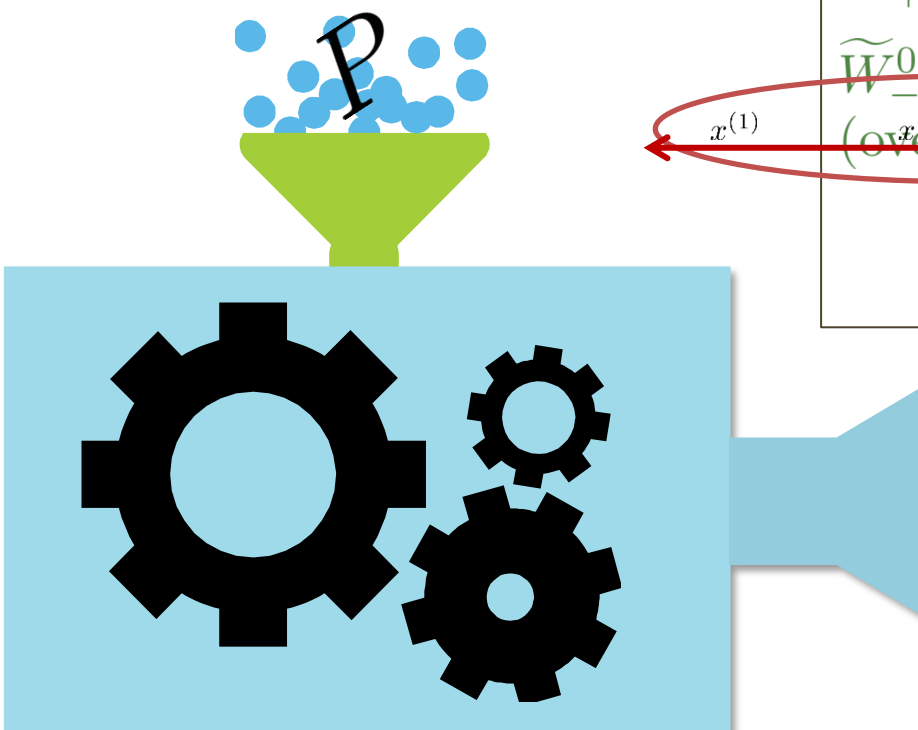
$W_+^1 = \max_{x:f(x)=1} w_+(x)$

$\widetilde{W}_-^0 = \max_{x:f(x)=0} \tilde{w}_-(x)$

(over min-error negative witnesses)

$$C_f = \sqrt{W_+^1 \widetilde{W}_-^0}\,\frac{1}{(1-\lambda)^{3/2}}$$





A quantum algorithm for f with query complexity C_f

Example: Gapped t-Threshold

$$f(x) = \begin{cases} 1 & \text{if } |x| \geq t \\ 0 & \text{if } |x| \leq t/2 \end{cases}$$

Recall from the span program for OR: $w_+(x) = \frac{1}{|x|}$ $w_-(x) = n$

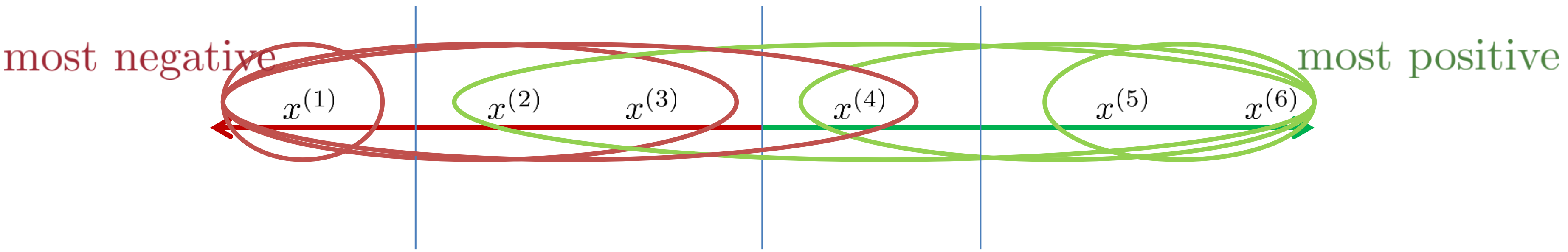
$$\frac{\max_{x:f(x)=1} w_+(x)}{\min_{x:f(x)=0} w_+(x)} = \frac{\max_{x:|x|\geq t} w_+(x)}{\min_{x:|x|\leq t/2} w_+(x)} = \frac{1}{2}$$

$$W_+^1 := \max_{x:f(x)=1} w_+(x) = \max_{x:|x|\geq t} \frac{1}{|x|} = \frac{1}{t}$$

$$\widetilde{W}_-^0 := \max_{x:f(x)=0} \tilde{w}_-(x) = n$$

$$C_f = \sqrt{W_+^1 W_-^0} = \sqrt{n/t}$$

Approximate Span Programs



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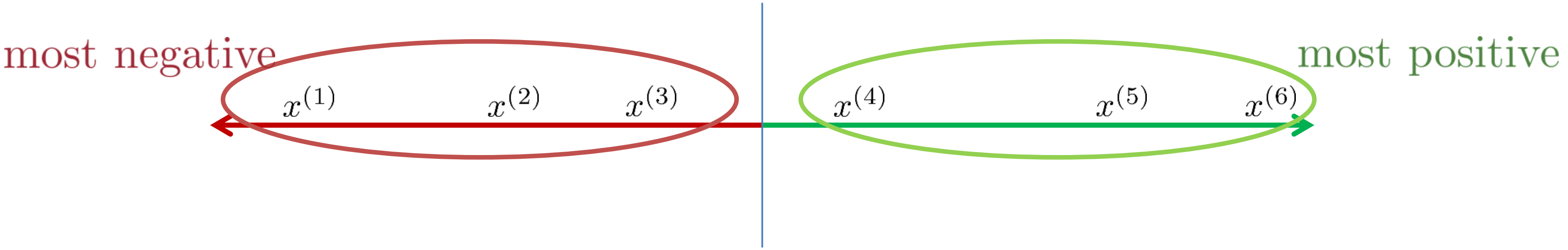

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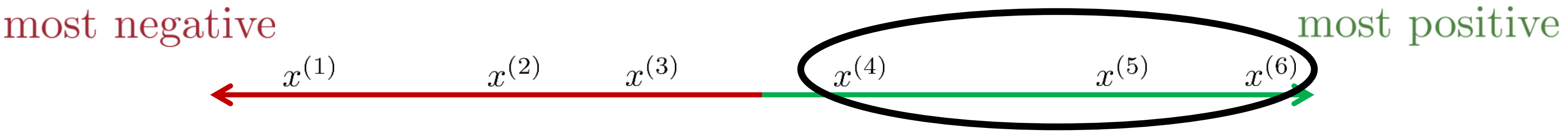

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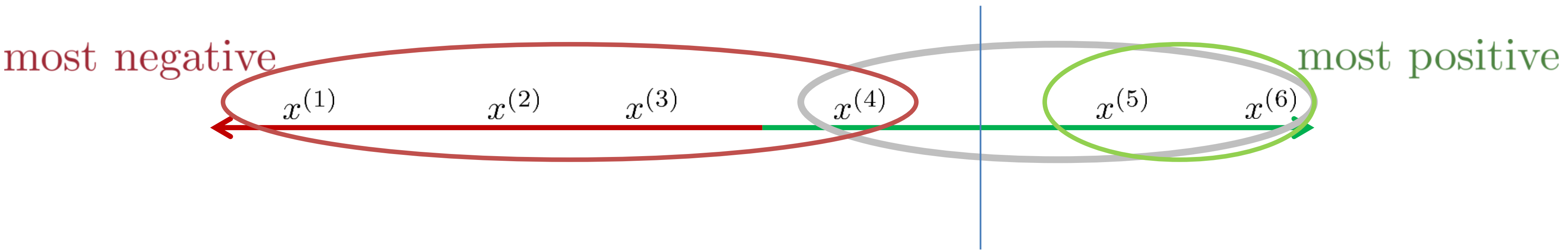

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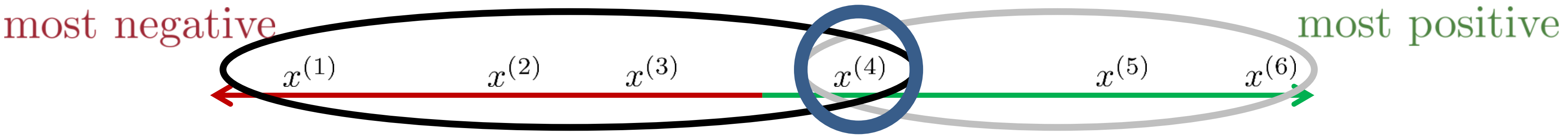

Estimating the Witness Size

Span program

Quantum algorithm for estimating $w_+(x)$ with query complexity $O\left(\frac{1}{\varepsilon^{3/2}}\sqrt{w_+(x)\widetilde{W}_-}\right)$

$\mathtt{Est}$ quantum algorithm for estimating $w_+(x)$

Example: Estimating $|x|$

Recall from the span program for OR: $w_+(x) = \frac{1}{|x|}$ $\widetilde{W}_-^0 = n$

Quantum query complexity of estimating $|x|$:

$$\frac{1}{\varepsilon^{3/2}}\sqrt{w_+(x)\widetilde{W}_-} = \frac{1}{\varepsilon^{3/2}}\sqrt{\frac{n}{|x|}}$$

not optimal

New Application: Effective Resistance

Input: G, a graph on $[n]$; $s,t \in [n]$

Effective resistance between s and t in G: $$R_{s,t}(G) = \min_{\theta} \sum_{e \in E(G)} \theta(e)^2$$ over unit flows with source s and sink t

Quantum query complexity of estimating $R_{s,t}$: $O\left(\frac{1}{\varepsilon^{3/2}} n \sqrt{R_{s,t}(G)}\right)$

Quantum query complexity est. $R_{s,t}$ when $\lambda_2(G) \geq \mu$: $O\left(\frac{1}{\varepsilon} n \sqrt{\frac{R_{s,t}(G)}{\mu}}\right)$

SUMMARY

- Span programs have been used to design quantum algorithms

- Span programs can be used to design quantum algorithms in two new ways:

  1) Larger class of decision problems for each span program

  2) Estimation problems

  - No longer restricted to decision problems
  - New applications to estimating effective resistance

- Further applications?

THANKS!