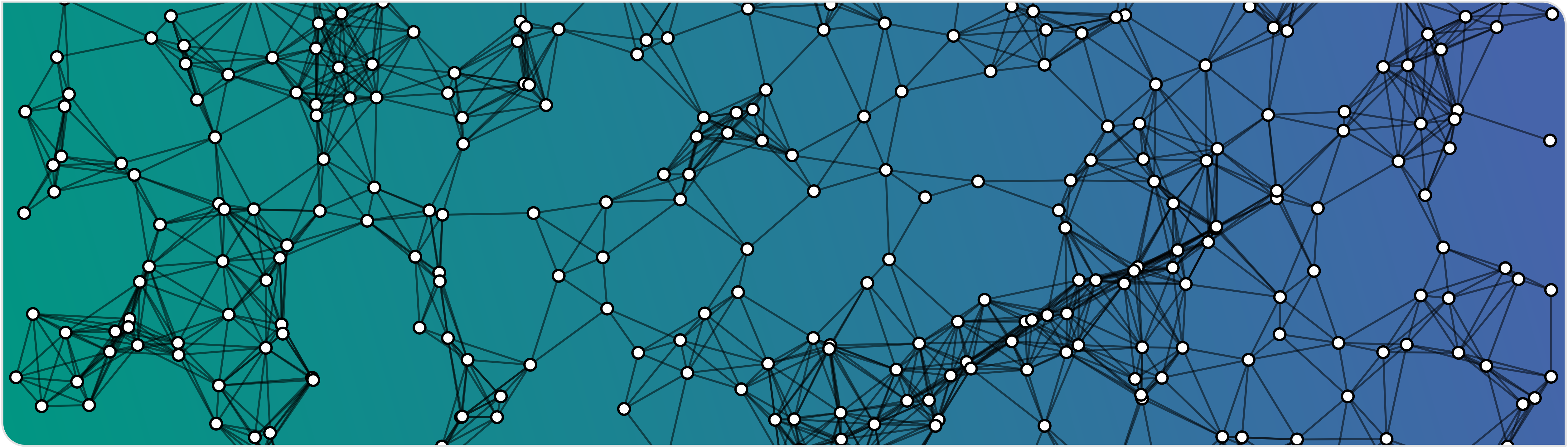
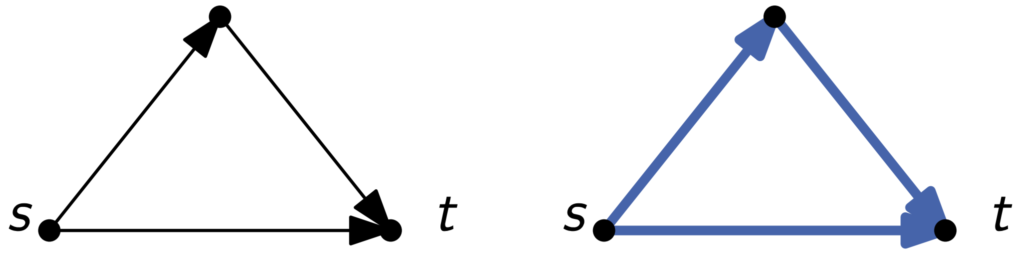
Dynamic Flows with Time-Dependent Capacities

Thomas Bläsius, **Adrian Feilhauer**, Jannik Westenfelder



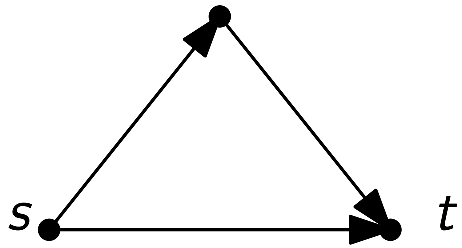
Dynamic Flows

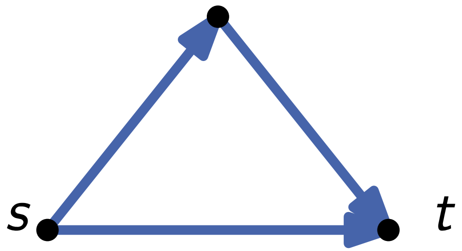
- Network flow from source s to target t

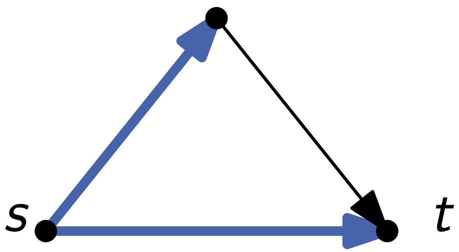


Dynamic Flows

- Network flow from source s to target t
- Traversing edges takes time







$\Theta \in [0, 1)$

Dynamic Flows

- Network flow from source s to target t
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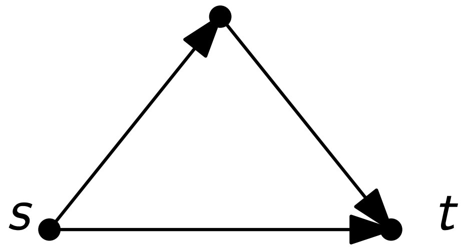
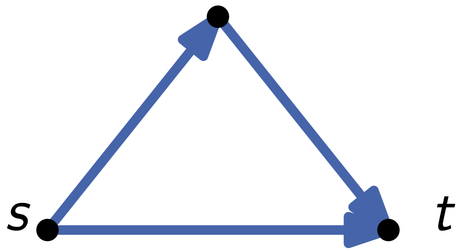
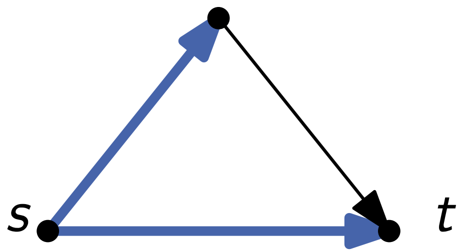
- Maximize flow reaching t before time horizon T







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$T = 4$

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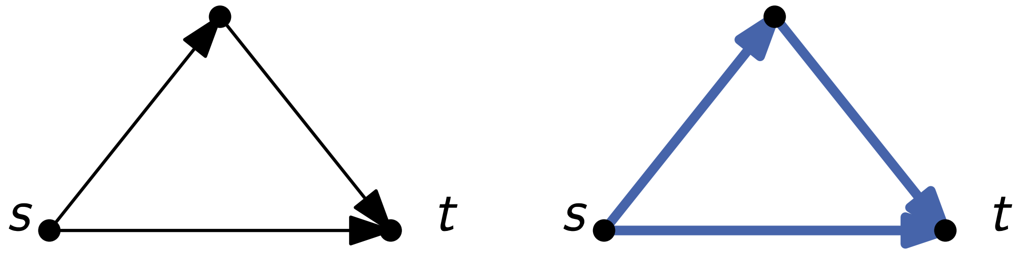

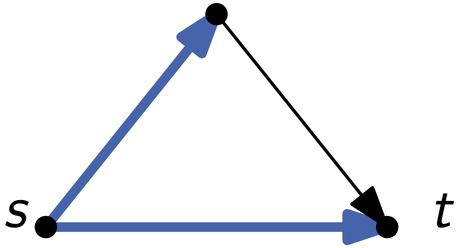

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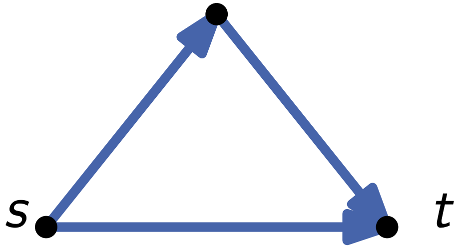

$\Theta \in [1, 2)$

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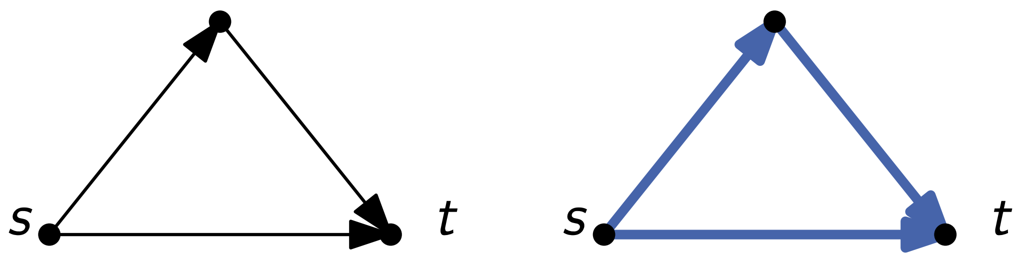

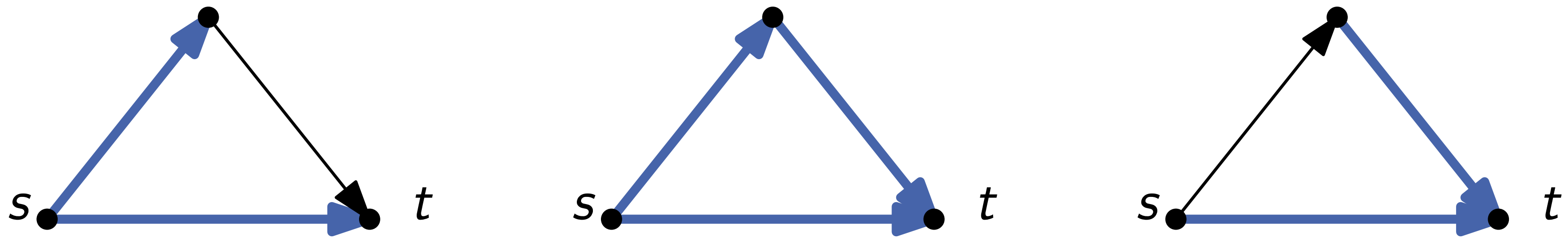

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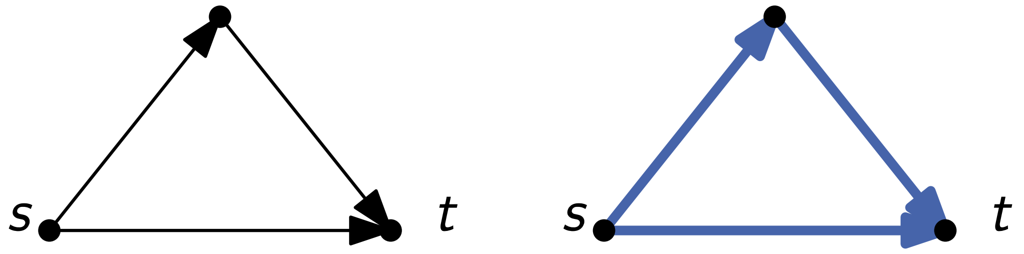

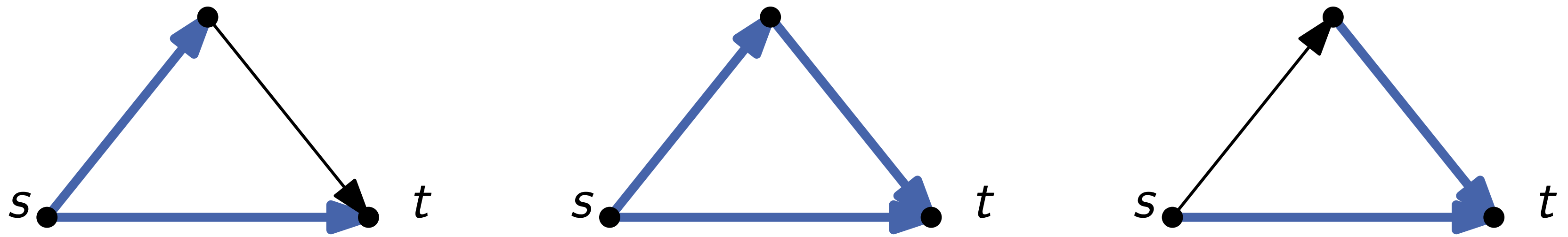

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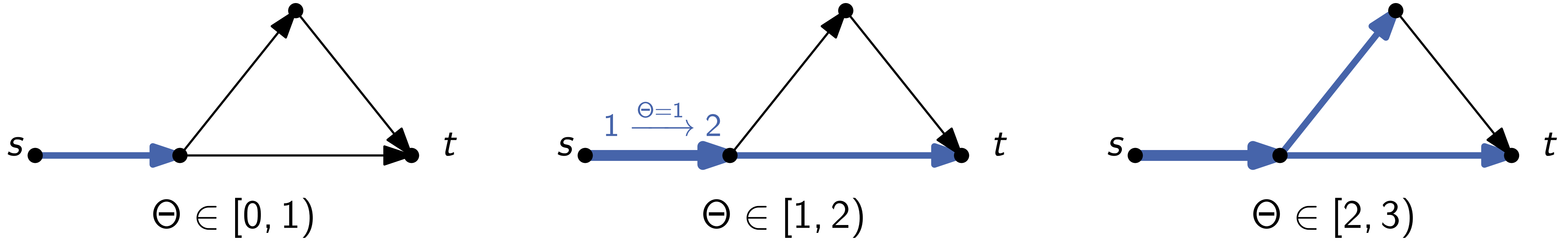
- Solvable via temporally repeated flows [Ford, Fulkerson 1958]

Time Dependent Capacities

- Capacities change over time (here piecewise constant)



- Solvable via time expanded networks

Computational Complexity

Theorem:
Finding maximum dynamic flows with time dependent capacities is weakly NP-hard.

- Reduction from PARTITION

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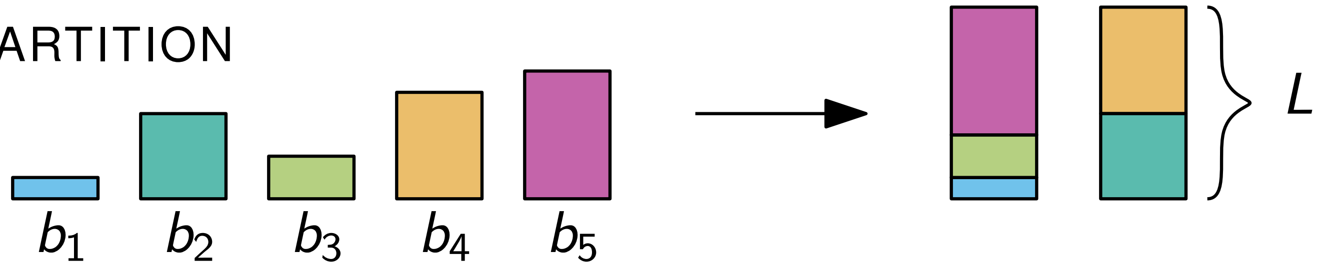

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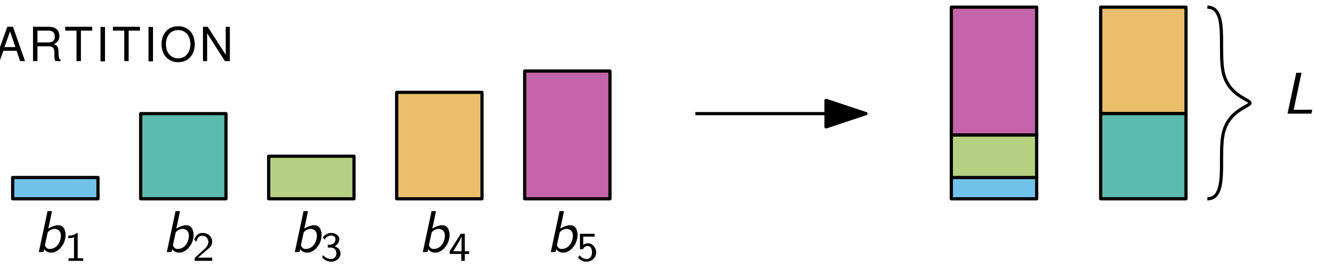
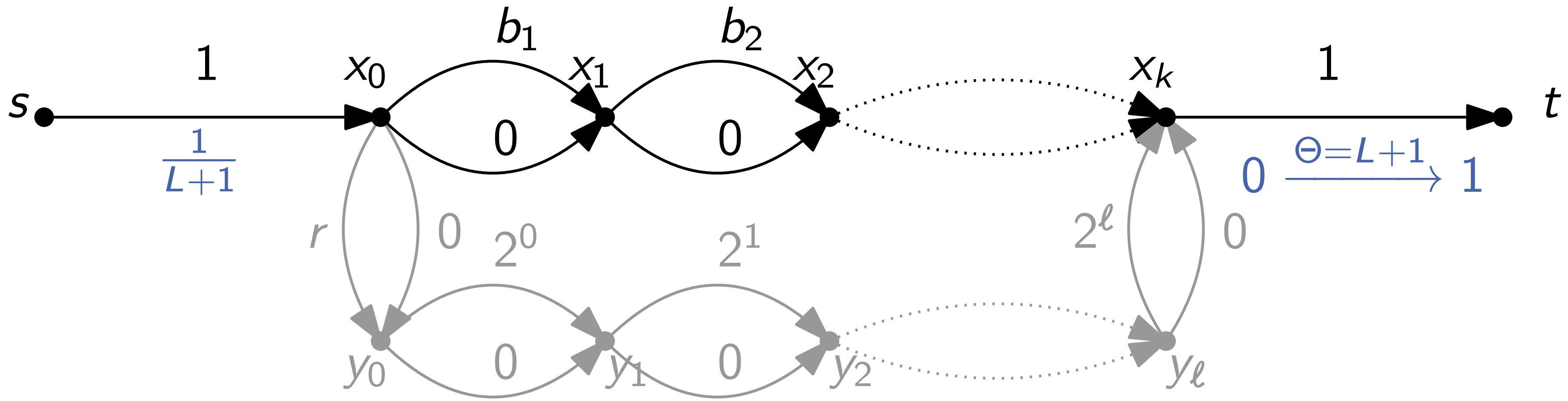


- Flow of value 1 is equivalent to the PARTITION instance being solvable



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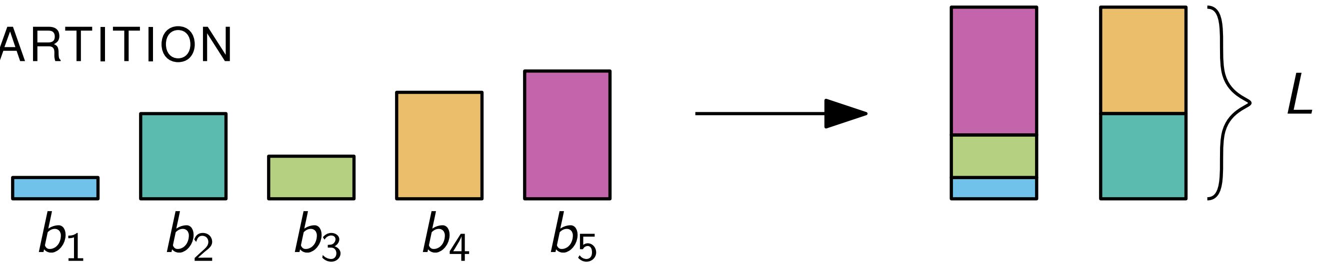

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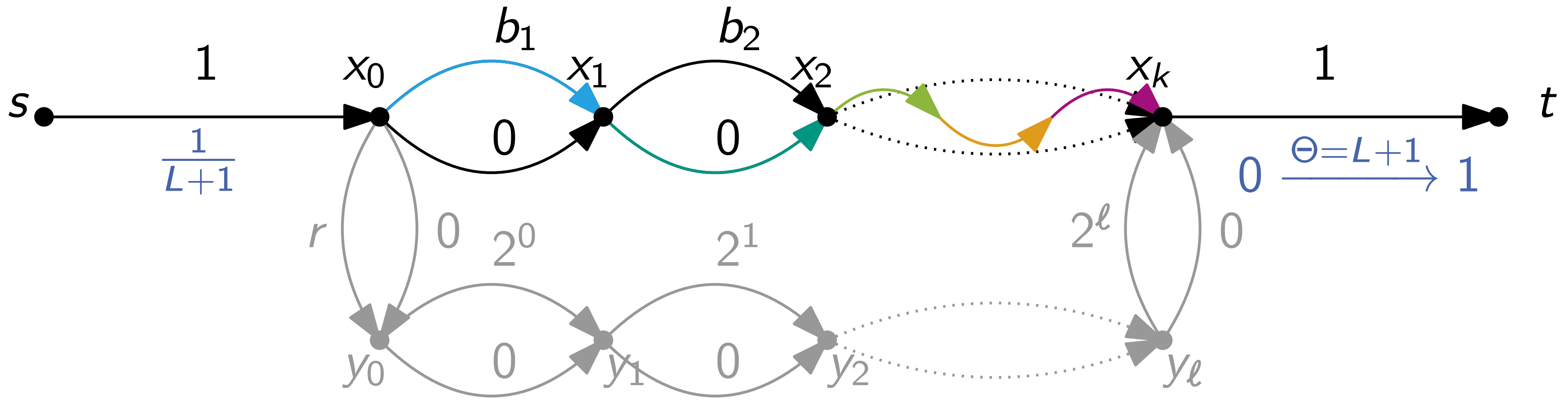

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Computational Complexity

- Pseudopolynomial solution via time expanded networks

	time-dependent capacities	time-dependent transit times
finite time horizon	hard for 1 change	
infinite considered time		

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Computational Complexity

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- Potentially unwanted effects near $T \rightarrow$ infinite considered time

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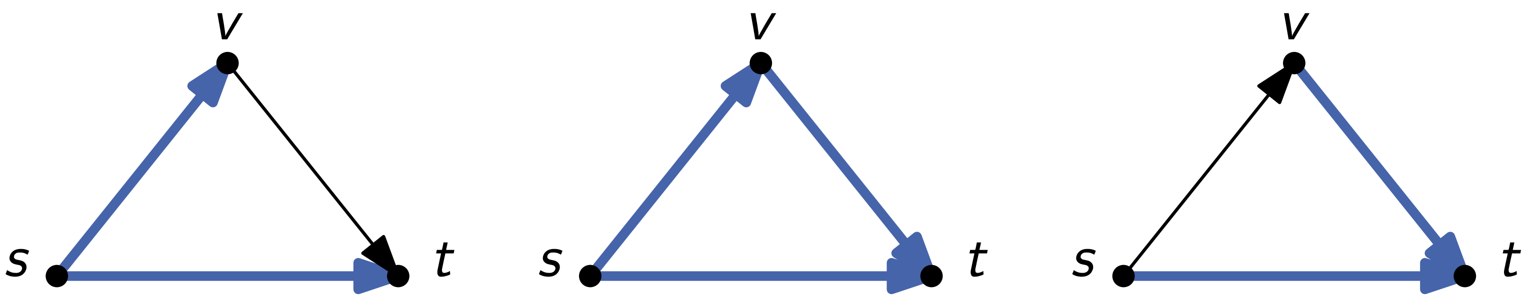
- Is finding maximum dynamic flows in NP?
- How complicated can those maximum flows become?

Dynamic Cut-Flow Duality

- Cuts are vertex partitions $S, \bar{S}$
- Cut capacity is possible flow from S to $\bar{S}$
- Dynamic Cuts can change over time
- Number of partition changes is the cut's *complexity*



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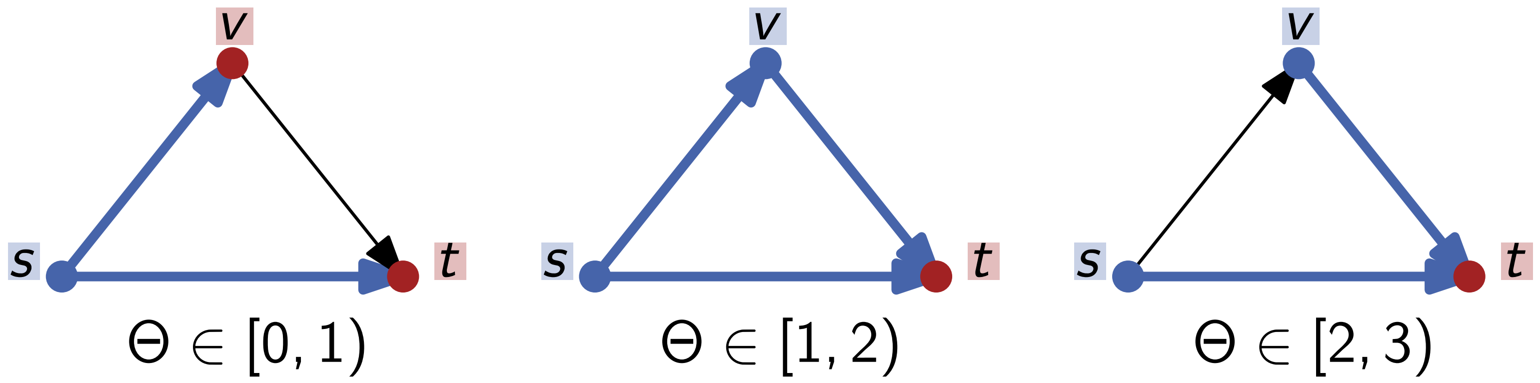

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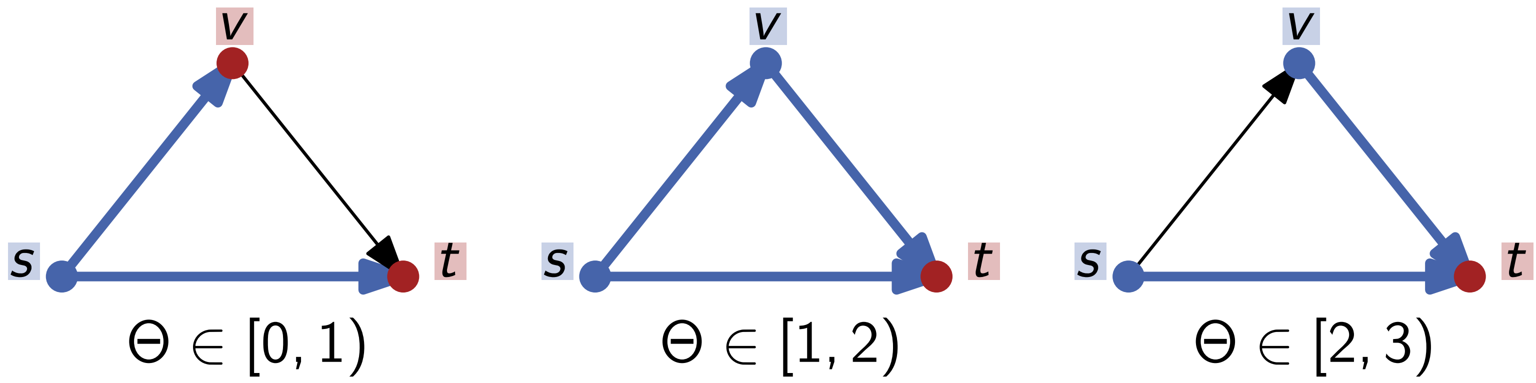
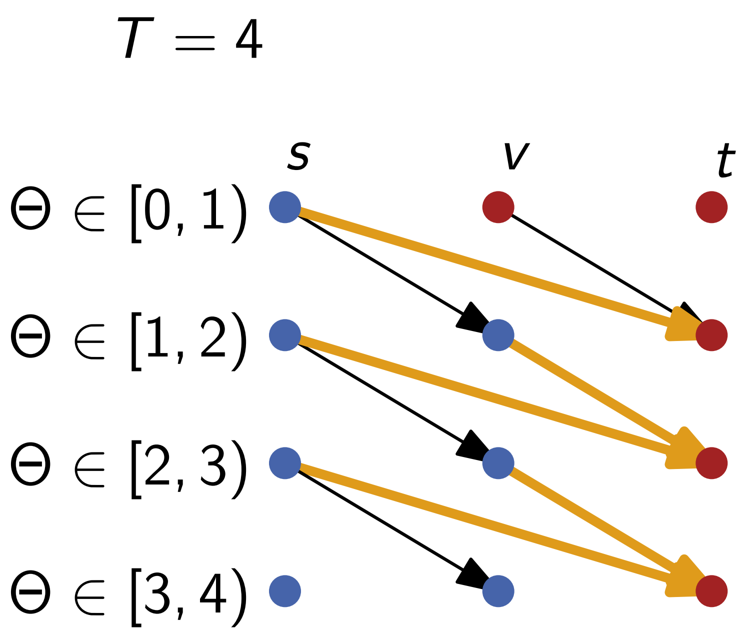
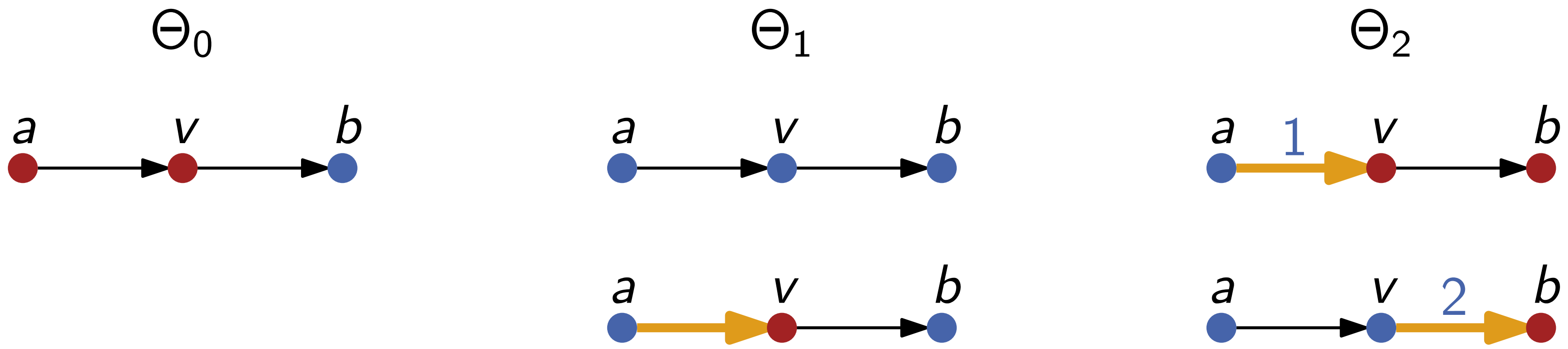




Theorem (Cut-Flow Duality [Philpott 1990]):
The capacity of a minimum dynamic cut equals the value of a maximum dynamic flow.

Mimicking Cuts

- How complex are minimum dynamic cuts?
- Partition changes can cascade (vertex v)



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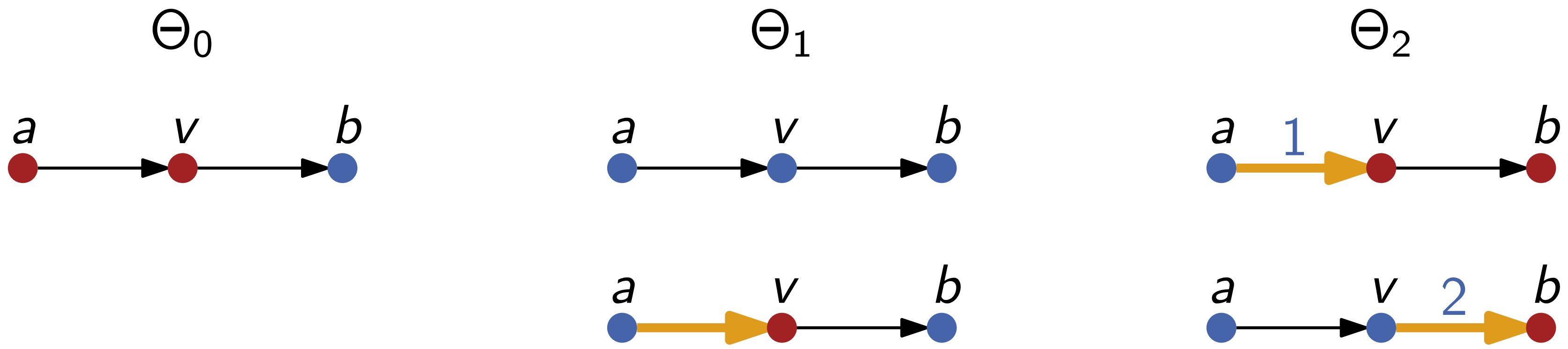

- **Idea:** use this to make vertex y mimic the behavior of x

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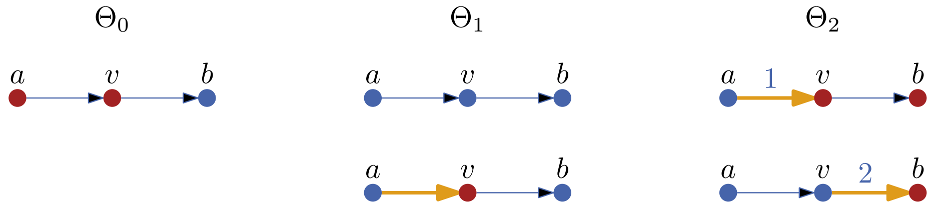

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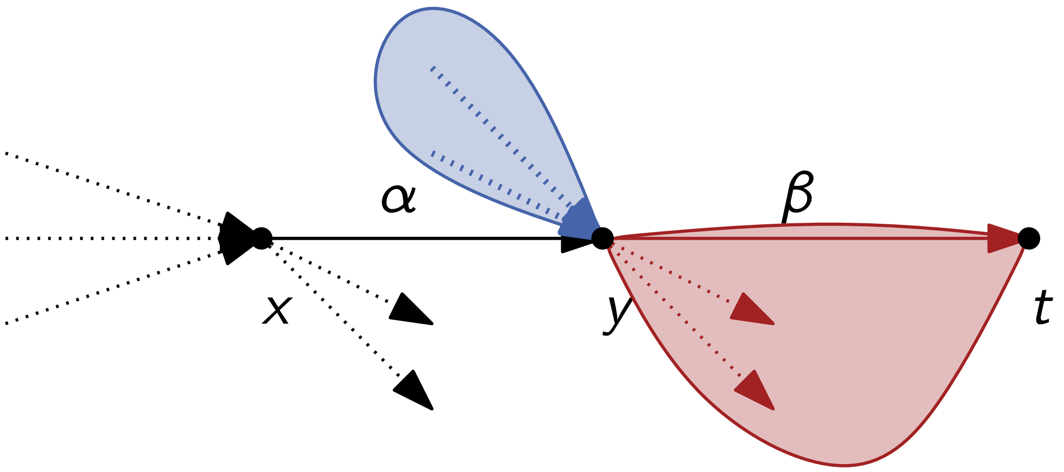
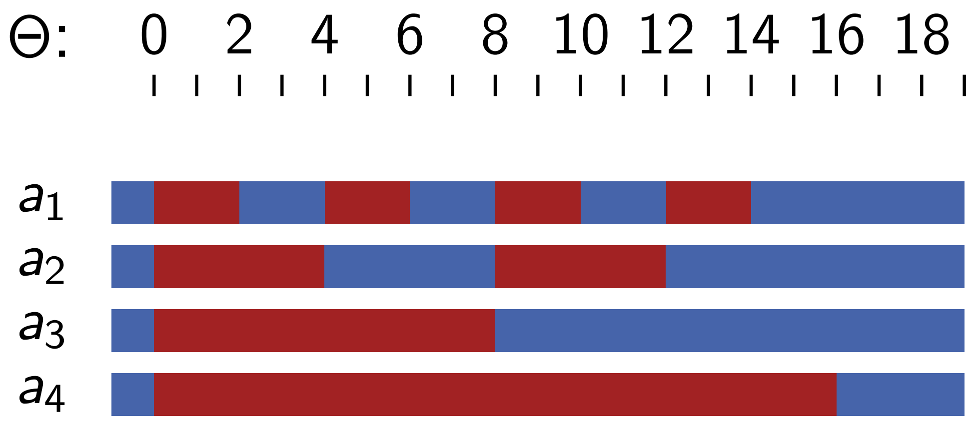
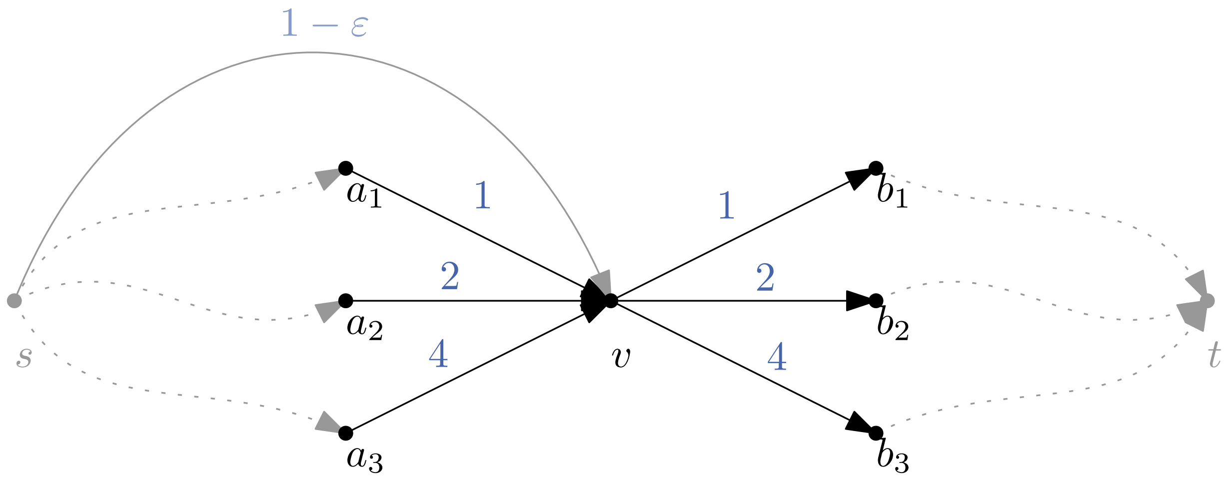


$$\alpha > \sum_{w:(y,w)\in E} u_{(y,w)}$$
$$\beta > \sum_{w:(w,y)\in E\setminus(x,y)} u_{(w,y)}$$

Exponentially Complex Cuts

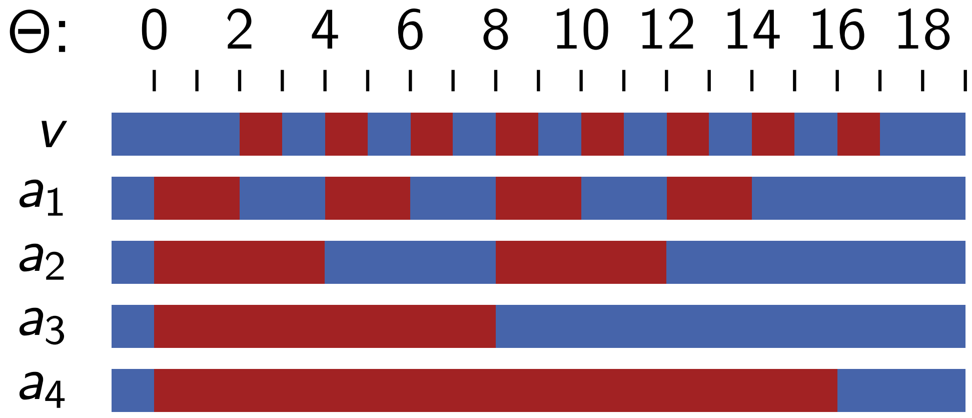
- Given partition changes of vertices a_i

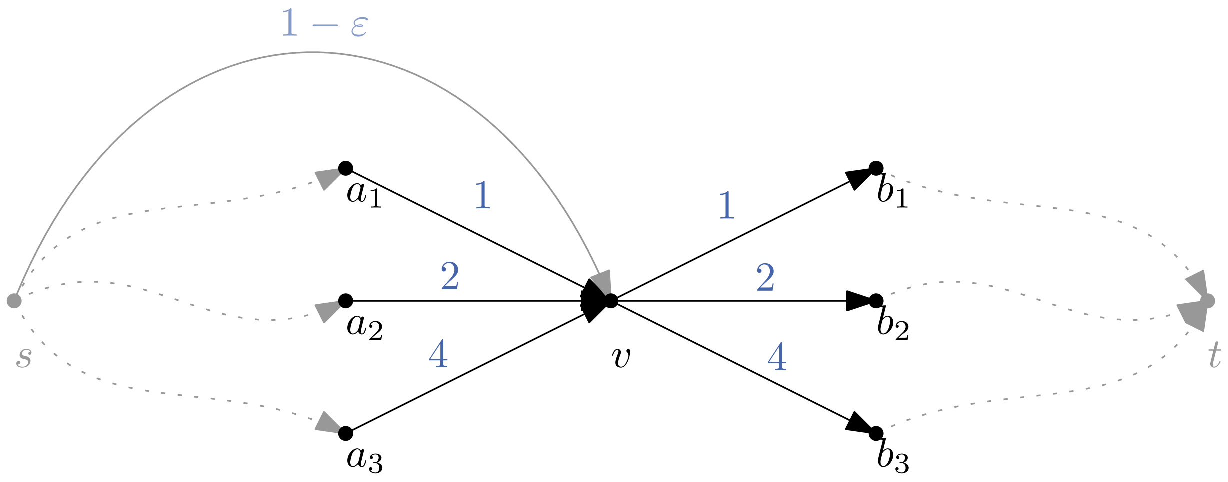




Exponentially Complex Cuts

- Given partition changes of vertices a_i
- More partition changes at vertex v





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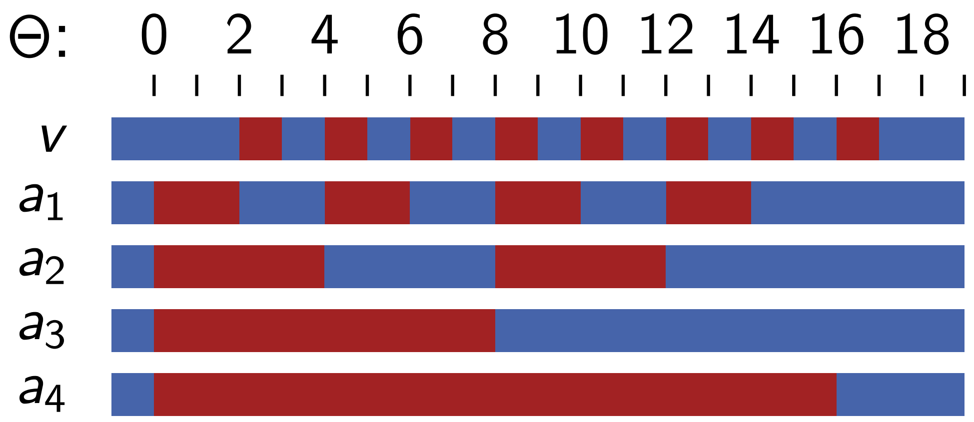
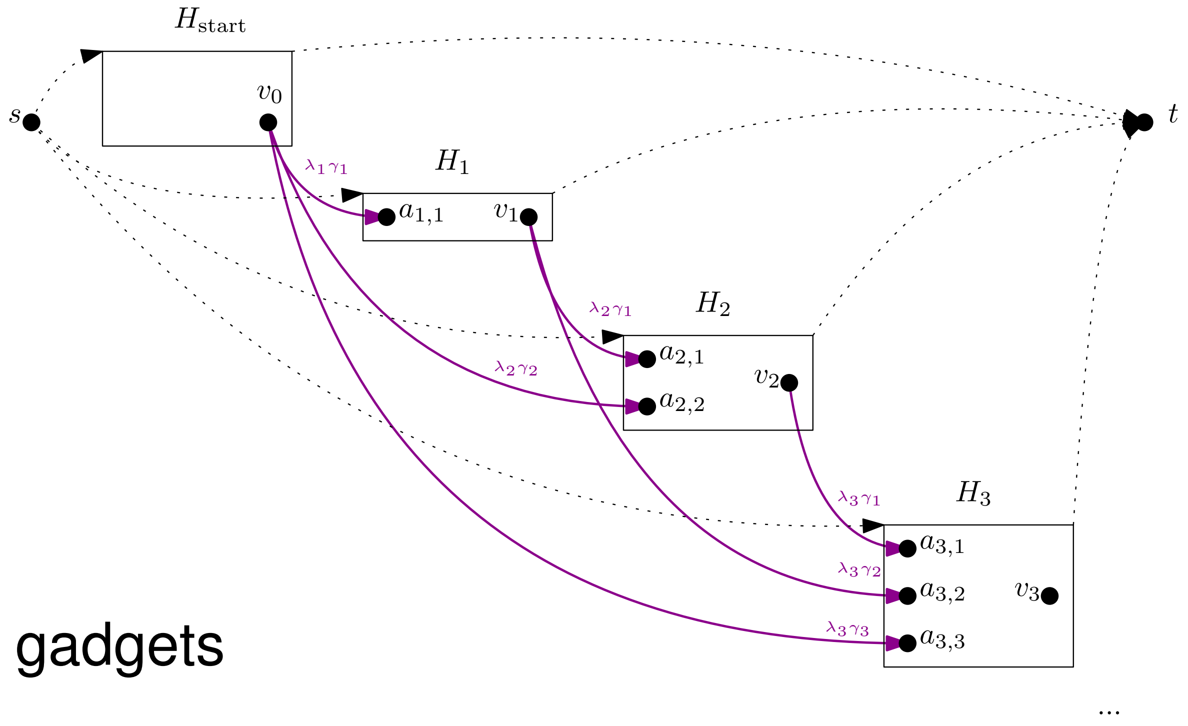


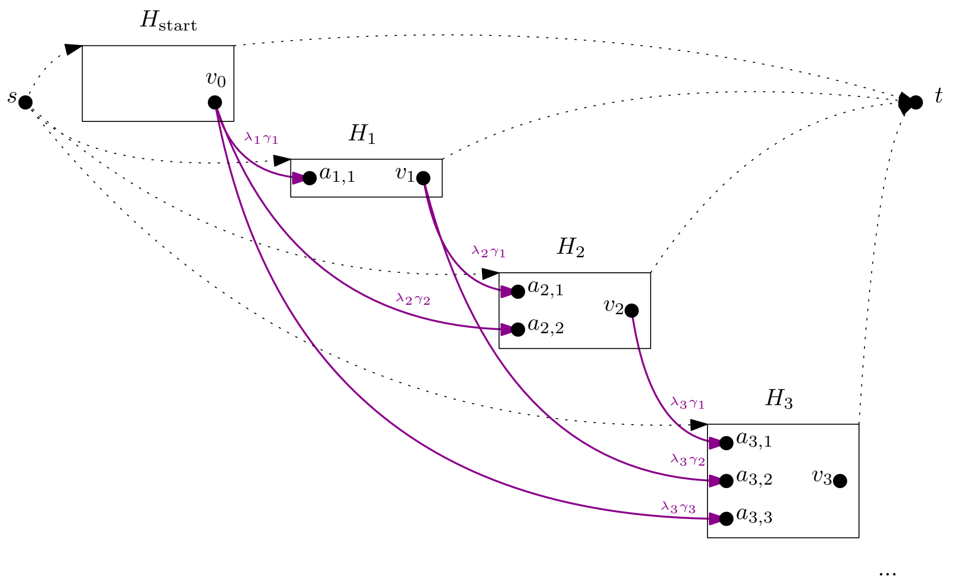


- Use smaller counters to provide a_i for larger gadgets

Theorem:
Minimum dynamic cuts may be exponentially complex.

Exponentially Complex Flows

- Cut and flow complexity can occur independently
  - Simple flow and complex cut
  - Simple cut and complex flow



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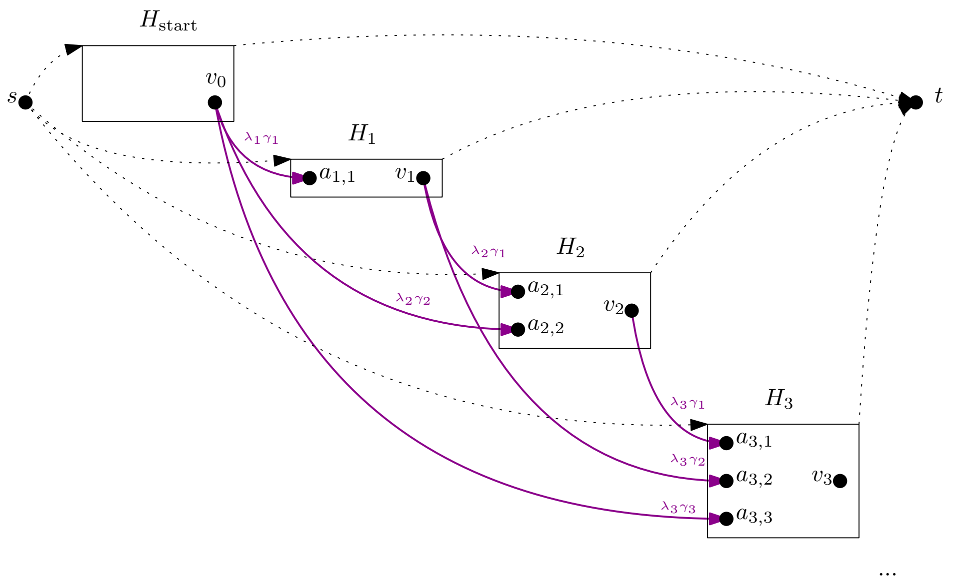
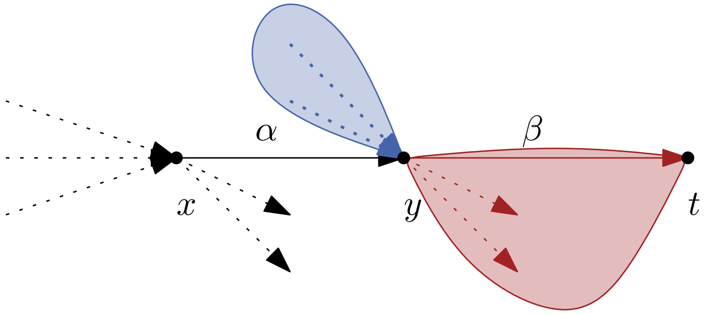
- Cut-Flow-Duality is still helpful:
- Previous construction has complex cut and flow



Theorem:
Maximum dynamic flows may be exponentially complex

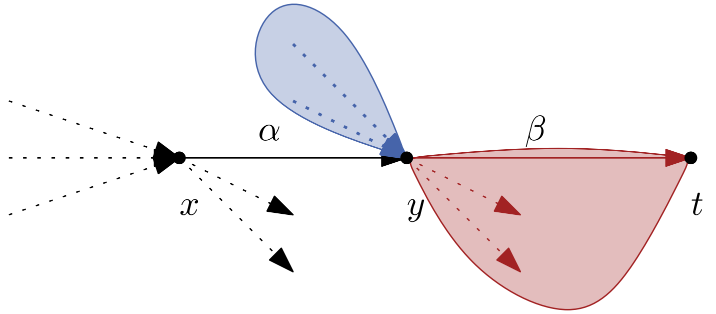
Complexity of Dynamic Flows and Cuts

- Gadgets provide structural insight into dynamic cuts



Complexity of Dynamic Flows and Cuts

- Gadgets provide structural insight into dynamic cuts
- Cut complexity aligns with computational complexity



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